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Journal of Geometry and Physics 57 (2007) 2077–2097

JOURNAL OF GEOMETRY AND PHYSICS

www.elsevier.com/locate/jgp

Self-dual Zollfrei conformal structures with α-surface foliation

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Received 30 December 2006; received in revised form 23 April 2007; accepted 12 May 2007
Available online 18 May 2007

Abstract

A global twistor correspondence is established for neutral self-dual conformal structures with α-surface foliation when the structure is close to the standard structure on $S^2 \times S^2$. We need to introduce some singularity for the α-surface foliation such that the leaves intersect on a fixed 2-sphere. In this correspondence, we prove that a natural double fibration is induced on some quotient spaces which is equal to the standard double fibration for the standard Zoll projective structure. We also give local general forms of neutral self-dual metrics with α-surface foliation.

JGP SC: Differential geometry; Spinors and twistors

MSC: 53C28; 32L25; 53C50

Keywords: Zollfrei metric; Twistor correspondence

1. Introduction

LeBrun and Mason investigated two kinds of twistor-type correspondences in [7,8]. One of them is the correspondence for the Zoll projective structure on two-dimensional manifolds [7]. A projective structure is an equivalence class of torsion-free connections under the projective equivalence, where two torsion-free connections are called projectively equivalent if they have exactly the same unparameterized geodesics. A projective structure is called Zoll when all the maximal geodesics are closed. LeBrun and Mason proved that there is a one-to-one correspondence between

- equivalence classes of orientable Zoll projective structures $(B, [\nabla])$, and
- equivalence classes of totally real embeddings $\iota : \mathbb{RP}^2 \to \mathbb{CP}^2$,

when they are close to the standard structures. Here B is identified with the moduli space of holomorphic disks on $\mathbb{CP}^2$ whose boundaries are contained in $N = \iota(\mathbb{RP}^2)$.

The second twistor correspondence constructed by LeBrun and Mason is the one for four-dimensional manifolds equipped with a neutral self-dual Zollfrei conformal structure [8]. An indefinite metric on a four-dimensional manifold is called neutral when the signature is $(++--)$, and here we consider the indefinite conformal structures represented by such metrics. For a neutral metric on a 4-manifold, we can define the self-duality condition like in the Riemannian

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doi:10.1016/j.geomphys.2007.05.004

case (cf. [4,5,8]). An indefinite metric is called Zollfrei when all the maximal null geodesics are closed. In the neutral four-dimensional case, the Zollfrei condition and the self-dual condition depend only on the conformal class [8]. LeBrun and Mason introduced the notion of space–time orientation for a 4-manifold with neutral metric, and proved that there is a one-to-one correspondence between

- equivalence classes of space–time oriented self-dual Zollfrei conformal structures $(M, [g])$, and
- equivalence classes of totally real embeddings $\iota : \mathbb{RP}^3 \to \mathbb{CP}^3$,

when they are close to the standard structures. Here M is identified with the moduli space of holomorphic disks on $\mathbb{CP}^3$ whose boundaries are contained in $P = \iota(\mathbb{RP}^3)$. They also proved that only $S^2 \times S^2$ admits a space–time oriented neutral self-dual Zollfrei conformal structure.

These two twistor correspondences are real, non-analytic and global versions of two of the three twistor correspondences explained in [6] by Hitchin. The three are twistor correspondences for (1) complex surfaces with projective structure, (2) complex 4-manifolds with anti-self-dual conformal structure and (3) complex 3-manifolds with Einstein–Weyl structure. The corresponding twistor space is given by complex manifolds Z with an embedded $\mathbb{CP}^1$ whose normal bundle is $\mathcal{O}(1)$, $\mathcal{O}(1) \oplus \mathcal{O}(1)$ or $\mathcal{O}(2)$ respectively. The geometric structures (1), (2) and (3) are given as natural structures on the moduli spaces of such an embedded $\mathbb{CP}^1$ in Z. Hitchin's argument is local, and the description is based on holomorphic category. The twistor space for (1) is sometimes called mini-twistor space (cf. [2]).

The twistor correspondence for (2) was originally discovered by Penrose [12], and the Riemannian version of this twistor correspondence is given by Atiyah, Hitchin and Singer [1]. In the Riemannian case, self-dual conformal structure is automatically analytic since the equation is elliptic. Moreover the family of rational curves on twistor space forms a globally defined foliation, and, for this reason, it is straightforward to translate the local description to the global case.

On the other hand, in the cases of LeBrun and Mason, the equations have non-analytic solutions in general, and the family of $\mathbb{CP}^1$ in the twistor space does not form a foliation different from the Riemannian case one. LeBrun and Mason overcame these difficulties by using two techniques: the first one is using the family of holomorphic disks instead of that of $\mathbb{CP}^1$, and the second one is setting in terms of the Zollfrei condition. Notice that the Zollfrei condition is an open condition in the space of neutral self-dual metrics (cf. [8]).

Recently, there has been some development concerning the reduction of the neutral self-dual conformal structures on 4-manifolds $(M, [g])$ (cf. [2,4,11]). Dunajski and West [4] proved that, if there is a null conformal Killing vector field on M, then there is a natural null surface foliation containing this Killing field, and that a natural projective structure is induced on the leaf space. Calderbank generalized this argument and weakened the assumption; the weakened assumption is given as a property for a null surface foliation on M. Both arguments are local, and formulated in a smooth category. They also studied the analytic case; then they showed that, under these conditions, a twistor correspondence of case (2) induces a twistor correspondence of case (1) as a reduction.

It would be natural to expect a theory of reduction for the two global twistor correspondences of LeBrun and Mason. The local theory of Dunajski, West and Calderbank would suggest that the natural class of such a theory is neutral self-dual Zollfrei conformal structures with closed null surface foliation. Even the standard conformal structure $(S^2 \times S^2, [g_0])$, however, is not contained in this class. Actually, on $(S^2 \times S^2, [g_0])$, any two closed α-surfaces intersect at exactly two points, so it is impossible to find a closed α-surface foliation, where the α-surface is one of the two kinds of null surfaces. The purpose of this paper is to set a nice class of neutral self-dual Zollfrei conformal structures equipped with an α-surface foliation with some singularity explained later. Then we prove that there is a one-to-one correspondence similar to the twistor correspondence of LeBrun and Mason, and that the reduction works globally.

In our situation, the induced projective structure on the leaf space is proved to be the standard Zoll projective structure. It would be an interesting problem to find some different formulations so that non-standard Zoll projective structures are induced by the reduction.

The organization of the paper is as follows. In Sections 2 and 3, we review the definitions and properties for projective structures and neutral self-dual conformal structures respectively. In particular in Section 3, we prepare an explicit description without using spinor calculus, which enables us to establish the general forms of neutral self-dual metrics with α-surface foliation in Section 4. In Section 5, we define a notion of basic α-surface foliation which we need to carry out the reduction. Calderbank defined the notion of self-dual α-surface foliation. In Appendix A, we show that basic is equivalent to self-dual under the assumption of self-duality for the metric. The basic foliation, however, rather fits to our description. By using this notion, we give a simple proof of the above results of Dunajski and West in Appendix B.

We treat the global situation in Sections 6 and 7. In Section 6, we formulate a class of neutral self-dual conformal structures with a suitable α-surface family, and we state the main theorem (Theorem 32). There is a low dimensional or mini-twistor version of the main theorem, and we prove this version in the rest of Section 6. The proof of the main theorem is presented in Section 7.

In this article, we follow LeBrun and Mason's conventions of orientations and the terminology of the α-surface and β-surface. We assume that all the manifolds and metrics are C^∞, and that the topology of maps between manifolds is C^∞-topology.

2. Projective structure

Let B be an oriented two-dimensional manifold, and let $\mathcal{W} = \mathbb{P}(TB \otimes \mathbb{C})$ and $\mathcal{W}_\mathbb{R} = \mathbb{P}(TB)$ be the projectivizations. Let $p : \mathcal{W} \to B$ and $p_\mathbb{R} : \mathcal{W}_\mathbb{R} \to B$ be the projections. Then every $w \in \mathcal{W} \setminus \mathcal{W}_\mathbb{R}$ corresponds to a complex line $L_w \subset T_bB \otimes \mathbb{C}$, where $b = p(w)$. Since $T_bB \otimes \mathbb{C} = L_w \oplus \bar{L}_w$, w defines a complex structure on T_bB. Let $\mathcal{W}_+^\circ$ be one of the two connected components of $\mathcal{W} \setminus \mathcal{W}_\mathbb{R}$ whose element defines an orientation preserving complex structure, and we put $\mathcal{W}_-^\circ$ as the other component. Let $\mathcal{W}_\pm$ be the closures of $\mathcal{W}_\pm^\circ$; then we have

$$\mathcal{W} = \mathcal{W}_+^\circ \cup \mathcal{W}_-^\circ \cup \mathcal{W}_\mathbb{R} = \mathcal{W}_+ \cup \mathcal{W}_-.$$

Let $V \subset B$ be a coordinate neighborhood with an oriented coordinate (y^0, y^1). By putting $\partial_i = \frac{\partial}{\partial y^i}$, we can trivialize $\mathcal{W} = \mathbb{P}(T_\mathbb{C}B)$ on $\mathcal{V}$ via

$$\mathbb{CP}^1 \times V \xrightarrow{\sim} \mathcal{W}|_V : ([\zeta_0 : \zeta_1], b) \longmapsto [\zeta_0\partial_0 + \zeta_1\partial_1]_b. \tag{1}$$

Notice that $\mathcal{W}_+|_V \simeq \{(\zeta, b) \in \mathbb{CP}^1 \times V : \operatorname{Im}\zeta \geq 0 \text{ or } \zeta = \infty\}$, where $\zeta = \zeta_1/\zeta_0$ is the fiber coordinate.

Let ∇ be a torsion-free connection on B; then the connection form respecting the coordinate (y^0, y^1) is given by the $\mathfrak{gl}(2, \mathbb{R})$-valued 1-form ω:

$$\omega = (\omega^i_j), \qquad \nabla\partial_j = \omega^i_j\partial_i.$$

The horizontal lift of a tangent vector $e \in T_bB$ at $\zeta^i\partial_i \in T_bB$ is

$$\tilde{e} = e - \omega^i_j\zeta^j\frac{\partial}{\partial\zeta^i}. \tag{2}$$

Projecting to $\mathbb{P}(TB)$, the horizontal lift of e on $\mathcal{W}$ at $\zeta = \zeta_1/\zeta_0$ is given by

$$\tilde{e} = e - \left(\omega^1_0 + \zeta(\omega^1_1 - \omega^0_0) - \zeta^2\omega^0_1\right)(e)\frac{\partial}{\partial\zeta}. \tag{3}$$

Now we define a rank 1 distribution $L_\mathbb{R}$ on $\mathcal{W}_\mathbb{R}$ as the tautological lifts, i.e. $L_{\mathbb{R},(x,\zeta)}$ is the horizontal lift of the tangent line $\langle\partial_0 + \zeta\partial_1\rangle$, where $x \in B$ and $\zeta \in \mathbb{RP}^1 \cong \mathcal{W}_{\mathbb{R},x}$ is the local fiber coordinate. From (3), we obtain $L_\mathbb{R} = \langle\mathfrak{n}\rangle$ where

$$\mathfrak{n} = \partial_0 + \zeta\partial_1 - \left(\omega^1_0 + \zeta(\omega^1_1 - \omega^0_0) - \zeta^2\omega^0_1\right)(\partial_0 + \zeta\partial_1)\frac{\partial}{\partial\zeta}. \tag{4}$$

We can define a complex distribution L on $\mathcal{W}_+$ by $L = \langle\mathfrak{n}\rangle$, where $\mathfrak{n}$ is extended to the vector field on $\mathcal{W}_+$ by the analytic continuation for $\zeta \in \mathbb{CP}^1$. By definition, we have $L|_{\mathcal{W}_\mathbb{R}} = L_\mathbb{R} \otimes \mathbb{C}$. If we put $K = L + \langle\frac{\partial}{\partial\bar{\zeta}}\rangle$, then K defines an almost complex structure on $\mathcal{W}_+ \setminus \mathcal{W}_\mathbb{R}$ since K satisfies $T\mathcal{W}_+ \otimes \mathbb{C} = K \oplus \bar{K}$ on $\mathcal{W}_+ \setminus \mathcal{W}_\mathbb{R}$.

Torsion-free connections ∇ and ∇' on B are called projectively equivalent if and only if they define exactly the same unparameterized geodesics. We call a projective structure for a projectively equivalent class $[\nabla]$.

Proposition 1 (*LeBrun and Mason [7]*).

(1) *L and K are defined only by* $[\nabla]$*,*
(2) *L and K are integrable.*

Definition 2. A projective structure $(B, [\nabla])$ is called Zoll if and only if all of the maximal geodesics on B are closed.

Theorem 3 (*LeBrun and Mason [7]*). *There is a one-to-one correspondence between*

- *equivalence classes of oriented Zoll projective structures* $(B, [\nabla])$, *and*
- *equivalence classes of totally real embeddings* $\iota : \mathbb{RP}^2 \to \mathbb{CP}^2$,

when they are close to the standard structures. The correspondence is characterized by a double fibration $B \xleftarrow{p} (\mathcal{W}_+, \mathcal{W}_{\mathbb{R}}) \xrightarrow{q} (\mathbb{CP}^2, N)$, *where* $N = \iota(\mathbb{RP}^2)$, p *is the projection, and* q *is a surjection which is holomorphic on the interior of* $\mathcal{W}_+$.

The rough sketch of the proof is the following. If $(B, [\nabla])$ is given, then we can construct $(\mathcal{W}_+, \mathcal{W}_{\mathbb{R}})$ equipped with a rank 1 foliation on $\mathcal{W}_{\mathbb{R}}$. Collapsing this foliation, we obtain the space $(\mathbb{CP}^2, N)$. Conversely, if ι is given, then there is a family of holomorphic disks in $\mathbb{CP}^2$ such that the boundaries of disks are contained in N and that this family defines a foliation on $\mathbb{CP}^2 \setminus N$. We also remark that each holomorphic disk in this family is characterized by the condition: the relative homology class of the disk generates $H_2(\mathbb{CP}^2, N) \cong \mathbb{Z}$. We define B to be the parameter space of this family. Then a Zoll projective structure $[\nabla]$ on B is induced so that each closed geodesic is written in the form $p \circ q^{-1}(\zeta)$ for some $\zeta \in N$. Notice that such a family of holomorphic disks is uniquely determined as a deformation of the standard family if ι is close enough to the standard embedding.

3. Neutral metric

Let M be an oriented four-dimensional manifold, and let g be a neutral metric on M where a neutral metric is an indefinite metric of split signature. An oriented local frame (e_0, e_1, e_2, e_3) of the tangent bundle TM is called a null tetrad if and only if its metric tensor $g_{\mu\nu} = g(e_\mu, e_\nu)$ is given by

$$g = (g_{\mu\nu}) = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix}. \tag{5}$$

Notice that, if (e_μ) is a null tetrad, then we obtain $g(\lambda, \lambda) = \det \begin{pmatrix} \lambda^0 & \lambda^2 \\ \lambda^1 & \lambda^3 \end{pmatrix}$ for a tangent vector $\lambda = \sum \lambda^\mu e_\mu$. When we make use of null tetrads, the structure group of TM reduces to the Lie group

$$SO(2,2) := \left\{ P \in SL(4, \mathbb{R}) : {}^t P\, g P = g \right\}. \tag{6}$$

$SO(2,2)$ has two connected components and we denote the identity component as $SO_0(2,2)$.

Definition 4 (*Cf. [8]*). M is called space–time orientable when the structure group of TM reduces to $SO_0(2,2)$.

Let $\mathrm{SL}(2,\mathbb{R})_+$ and $\mathrm{SL}(2,\mathbb{R})_-$ be copies of $\mathrm{SL}(2,\mathbb{R})$. For each $(A, B) \in \mathrm{SL}(2,\mathbb{R})_+ \times \mathrm{SL}(2,\mathbb{R})_-$, the transformation

$$\begin{pmatrix} e_0 & e_2 \\ e_1 & e_3 \end{pmatrix} \longmapsto A \begin{pmatrix} e_0 & e_2 \\ e_1 & e_3 \end{pmatrix} {}^t B$$

defines an element of $SO_0(2,2)$. In this way, we obtain a double covering $\mathrm{SL}(2,\mathbb{R})_+ \times \mathrm{SL}(2,\mathbb{R})_- \to SO_0(2,2)$. The corresponding Lie algebra isomorphism $\mathfrak{o}(2,2) \simeq \mathfrak{sl}(2,\mathbb{R})_+ \oplus \mathfrak{sl}(2,\mathbb{R})_-$ is given by

$$\begin{pmatrix} a & b & e & 0 \\ c & d & 0 & e \\ f & 0 & -d & b \\ 0 & f & c & -a \end{pmatrix} \longmapsto \begin{pmatrix} \dfrac{a-d}{2} & b \\ c & \dfrac{d-a}{2} \end{pmatrix} \oplus \begin{pmatrix} \dfrac{a+d}{2} & e \\ f & -\dfrac{a+d}{2} \end{pmatrix}. \tag{7}$$

Taking M smaller, we can assume that M is space–time oriented and the structure group of TM lifts to $\mathrm{SL}(2,\mathbb{R})_+ \times \mathrm{SL}(2,\mathbb{R})_-$. Then we obtain a decomposition $TM = S^+ \otimes S^-$, and the Levi-Civita connection ∇ on M induces the connections $\nabla^\pm$ on $S^\pm$. $S^\pm$ are called the positive and negative spin bundles, and $\nabla^\pm$ are called spin connections. If we take a local null tetrad (e_μ) on TM, then ∇ is represented by the connection form ω, where ω is a $\mathfrak{o}(2,2)$-valued 1-form, and the connection forms $\omega^\pm$ of $\nabla^\pm$ are $\mathfrak{sl}(2,\mathbb{R})_\pm$-valued 1-forms, which are defined as the components of the decomposition of ω by (7).

There is an eigenspace decomposition $\wedge^2 = \wedge^+ \oplus \wedge^-$ with respect to Hodge's $*$-operator, where $\wedge^2 = \wedge^2 TM$ and $\wedge^\pm$ are the eigenspaces for the eigenvalues ± 1. Let $\mathcal{V}$ be a null 2-plane in T_xM and v_1, v_2 be the basis of $\mathcal{V}$; then the bivector $v_1 \wedge v_2$ belongs to $\wedge^+$ or $\wedge^-$.

Definition 5. Let $\mathcal{V} = \langle v_1, v_2 \rangle \subset T_xM$ be a null 2-plane. $\mathcal{V}$ is called an α-plane if $v_1 \wedge v_2 \in \wedge^+$, and $\mathcal{V}$ is called a β-plane if $v_1 \wedge v_2 \in \wedge^-$. Let $S \subset M$ be an embedded surface and suppose that S is totally null, i.e. $T_xS \subset T_xM$ is null for every $x \in S$. S is called an α-surface if T_xS is an α-plane for every $x \in S$. A β-surface is defined in a similar way.

Let (M, g) be a space–time oriented neutral manifold, and (e_μ) be a null tetrad on an open set $U \subset M$. From now on, we define $e_2 = \phi_0$, $e_3 = \phi_1$ for later convenience. The following lemma is checked by a direct calculation.

Lemma 6. $\wedge^+ = \langle \varphi_1, \varphi_2, \varphi_3 \rangle$, $\wedge^- = \langle \psi_1, \psi_2, \psi_3 \rangle$, *where*

$$\begin{aligned} &\varphi_1 = e_0 \wedge e_1, \qquad \varphi_2 = \phi_0 \wedge \phi_1, \qquad \varphi_3 = \frac{1}{\sqrt{2}}(e_0 \wedge \phi_1 - e_1 \wedge \phi_0), \\ &\psi_1 = e_0 \wedge \phi_0, \qquad \psi_2 = e_1 \wedge \phi_1, \qquad \psi_3 = \frac{1}{\sqrt{2}}(e_0 \wedge \phi_1 + e_1 \wedge \phi_0). \end{aligned} \tag{8}$$

The neutral metric g induces indefinite metrics on $\wedge^\pm$ whose metric tensors are both given by the following matrix with respect to the frames (8):

$$h = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{9}$$

Let $\mathfrak{h}$ be a Lie algebra defined by

$$\mathfrak{h} = \left\{ X \in \mathfrak{gl}(3,\mathbb{R}) : {}^tX h + hX = 0 \right\} = \left\{ \begin{pmatrix} a & 0 & c \\ 0 & -a & b \\ b & c & 0 \end{pmatrix} \right\}.$$

The Levi-Civita connection ∇ induces connections on $\wedge^\pm$ whose connection forms are represented by an $\mathfrak{h}$-valued 1-form with respect to the frames (8).

We can check that the exterior product representation associated with $\wedge^-$ is

$$\rho^- : \begin{pmatrix} a & b & e & 0 \\ c & d & 0 & e \\ f & 0 & -d & b \\ 0 & f & c & -a \end{pmatrix} \longmapsto \begin{pmatrix} a-d & 0 & \sqrt{2}c \\ 0 & d-a & \sqrt{2}b \\ \sqrt{2}b & \sqrt{2}c & 0 \end{pmatrix}. \tag{10}$$

So the connection form of the induced connection on $\wedge^-$ is given by the $\mathfrak{h}$-valued 1-form $\theta = (\theta^k_l) = \rho^-(\omega)$, where ω is the connection form of the Levi-Civita connection. This connection naturally induces the connection on $\wedge^-_{\mathbb{C}}$, where $\wedge^-_{\mathbb{C}} = \wedge^- \otimes \mathbb{C}$ is the complexification. The horizontal lift of a tangent vector e on M at $\lambda^k \psi_k \in \wedge^-_{\mathbb{C}}$ is

$$\tilde{e} = e - \theta^k_l(e) \lambda^l \frac{\partial}{\partial \lambda^k}. \tag{11}$$

Let $\mathcal{Z} = \{[\psi] \in \mathbb{P}(\wedge^-_{\mathbb{C}}) : g(\psi, \psi) = 0\}$ and $\mathcal{Z}_{\mathbb{R}} = \{[\psi] \in \mathbb{P}(\wedge^-) : g(\psi, \psi) = 0\}$. Let $p : \mathcal{Z} \to M$ and $p_{\mathbb{R}} : \mathcal{Z}_{\mathbb{R}} \to M$ be the projections. Then a trivialization of $\mathcal{Z}$ on the open set $U \subset M$ is given by

$$\iota : \mathbb{CP}^1 \times U \xrightarrow{\sim} \mathcal{Z}|_U : ([\zeta_0 : \zeta_1], x) \longmapsto [\zeta_0^2 \psi_1 + \zeta_1^2 \psi_2 + \sqrt{2}\zeta_0\zeta_1\psi_3\,]_x. \tag{12}$$

This is nothing but the correspondence between the fiber coordinate $[\zeta_0 : \zeta_1] \in \mathbb{CP}^1$ and the complex β-plane $\langle \zeta_0 e_0 + \zeta_1 e_1,\ \zeta_0\phi_0 + \zeta_1\phi_1\rangle$, since we have

$$(\zeta_0 e_0 + \zeta_1 e_1) \wedge (\zeta_0\phi_0 + \zeta_1\phi_1) = \zeta_0^2\psi_1 + \zeta_1^2\psi_2 + \sqrt{2}\zeta_0\zeta_1\psi_3.$$

Restricting the fiber coordinate $[\zeta_0 : \zeta_1]$ to $\mathbb{RP}^1$, we also obtain a trivialization of $\mathcal{Z}_\mathbb{R}$, and each point in $\mathcal{Z}_\mathbb{R}$ corresponds to a real β-plane in the same manner.

Let $\beta_u \subset T_x M \otimes \mathbb{C}$ be the complex β-plane corresponding to $u \in \mathcal{Z} \setminus \mathcal{Z}_\mathbb{R}$, where $x = p(u)$. Since $T_x M \otimes \mathbb{C} = \beta_u \oplus \bar{\beta}_u$, z defines a complex structure J on $T_x M$, and it is easy to check that J preserves the metric g. Let $\mathcal{Z}_+^\circ$ be one of the two connected components of $\mathcal{Z} \setminus \mathcal{Z}_\mathbb{R}$ whose element defines an orientation preserving complex structure, and we put $\mathcal{Z}_-^\circ$ as the other component. Let $\mathcal{Z}_\pm$ be the closures of $\mathcal{Z}_\pm^\circ$; then we have

$$\mathcal{Z} = \mathcal{Z}_+^\circ \cup \mathcal{Z}_-^\circ \cup \mathcal{Z}_\mathbb{R} = \mathcal{Z}_+ \cup \mathcal{Z}_-.$$

Let $\wedge_\mathbb{C}^- \xrightarrow{\pi} \mathbb{P}(\wedge_\mathbb{C}^-)$ be the projectivization; then we obtain, at $(\lambda^1, \lambda^2, \lambda^3) = (\zeta_0^2, \zeta_1^2, \sqrt{2}\zeta_0\zeta_1)$,

$$\pi_*\left(\theta_l^k \lambda^l \frac{\partial}{\partial\lambda^k}\right) = \left(b + \zeta(d-a) - \zeta^2 c\right)\iota_*\left(\frac{\partial}{\partial\zeta}\right), \tag{13}$$

where $\zeta = \zeta_1/\zeta_0$ is the non-homogeneous coordinate. From (11), the horizontal lift of the tangent vector e on M to $\mathcal{Z}$ is

$$\tilde{e} = e - \left(b + \zeta(d-a) - \zeta^2 c\right)(e)\frac{\partial}{\partial\zeta}. \tag{14}$$

We can define a rank 2 distribution $E_\mathbb{R}$ on $\mathcal{Z}_\mathbb{R}$ as the tautological lifts, i.e. $E_{\mathbb{R},(x,\zeta)}$ is the horizontal lift of the β-plane $\langle e_0 + \zeta e_1, \phi_0 + \zeta\phi_1\rangle$, where $x \in M$ and $\zeta \in \mathbb{RP}^1 \cong \mathcal{Z}_{\mathbb{R},x}$. $E_\mathbb{R}$ is called the twistor distribution [4] or the Lax distribution [2]. From (14), we obtain $E_\mathbb{R} = \langle \mathfrak{m}_1, \mathfrak{m}_2\rangle$ where

$$\begin{aligned} \mathfrak{m}_1 &= e_0 + \zeta e_1 + Q_1(\zeta)\partial_\zeta, \qquad Q_1(\zeta) = -(b + \zeta(d-a) - \zeta^2 c)(e_0 + \zeta e_1),\\ \mathfrak{m}_2 &= \phi_0 + \zeta\phi_1 + Q_2(\zeta)\partial_\zeta, \qquad Q_2(\zeta) = -(b + \zeta(d-a) - \zeta^2 c)(\phi_0 + \zeta\phi_1). \end{aligned} \tag{15}$$

We can define a complex distribution E on $\mathcal{Z}_+$ by $E = \langle \mathfrak{m}_1, \mathfrak{m}_2\rangle$, where $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are extended to the vector fields on $\mathcal{Z}_+$ analytically for $\zeta \in \mathbb{CP}^1$. By definition, we have $E|_{\mathcal{Z}_\mathbb{R}} = E_\mathbb{R} \otimes \mathbb{C}$. If we put $D = E + \langle \frac{\partial}{\partial\bar{\zeta}}\rangle$, then D defines an almost complex structure on $\mathcal{Z}_+ \setminus \mathcal{Z}_\mathbb{R}$ since $T\mathcal{Z}_+ \otimes \mathbb{C} = D \oplus \bar{D}$ on $\mathcal{Z}_+ \setminus \mathcal{Z}_\mathbb{R}$. The following theorem is basic and proved in [8], and see also [4].

Theorem 7. (1) *E and D are defined only by the conformal class* $[g]$.

(2) *$E_\mathbb{R}$ is Frobenius integrable if and only if $[g]$ is self-dual. Moreover, the almost complex structure on $\mathcal{Z}_+ \setminus \mathcal{Z}_\mathbb{R}$ defined from D is integrable if and only if $[g]$ is self-dual.*

Definition 8. Let $(M, [g])$ be a neutral self-dual conformal structure; then $(M, [g])$ is called Zollfrei if and only if all of the maximal null geodesics on M are closed.

Theorem 9 (*LeBrun and Mason [8]*). *There is a one-to-one correspondence between*

- *equivalence classes of space–time oriented self-dual Zollfrei conformal structures $(M, [g])$, and*
- *equivalence classes of totally real embeddings $\iota : \mathbb{RP}^3 \to \mathbb{CP}^3$,*

when they are close to the standard structures. The correspondence is characterized by a double fibration $M \xleftarrow{p} (\mathcal{Z}_+, \mathcal{Z}_\mathbb{R}) \xrightarrow{q} (\mathbb{CP}^3, P)$, where $P = \iota(\mathbb{RP}^3)$, p is the projection, and q is a surjection which is holomorphic on the interior of $\mathcal{Z}_+$.

The proof is conceptually similar to that of Theorem 3. M is defined from ι as the parameter space of the family of holomorphic disks in $(\mathbb{CP}^3, P)$ foliating $\mathbb{CP}^3 \setminus P$. Such a family is uniquely determined if ι is close enough to the standard embedding.

4. α-surface foliation

Let (M, g) be a 4-manifold with a neutral metric, and we now suppose that there is a smooth α-surface foliation $\varpi : M \to B$, i.e. ϖ is a smooth surjection to a 2-manifold B such that each fiber $\varpi^{-1}(b)$ on $b \in B$ is an α-surface. Let $x \in M$ and $b = \varpi(x) \in B$; then we can take a local coordinate (x^0, x^1, x^2, x^3) around x and a coordinate (y^0, y^1) around b so that $(x^0, x^1, x^2, x^3) \overset{\varpi}{\mapsto} (y^0, y^1) = (x^0, x^1)$. Let $\mathcal{V} = \langle \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \rangle$ be the vertical distribution, and we use the notation $\partial_{x^\mu} = \frac{\partial}{\partial x^\mu}$ and so on.

Proposition 10. *There is a null tetrad $(e_0, e_1, \phi_0, \phi_1)$ on TM which satisfies*

(1) $e_0 = \partial_{x^0} + \alpha_0$ *and* $e_1 = \partial_{x^1} + \alpha_1$ *for some vertical vector fields* $\alpha_0, \alpha_1 \in \Gamma(\mathcal{V})$*,*
(2) ϕ_0 *and* ϕ_1 *are vertical, i.e.* $\phi_0, \phi_1 \in \Gamma(\mathcal{V})$*.*

Proof. We take e_0 and e_1 as follows. Let $\mathcal{V}'$ be an α-plane distribution which is transverse to $\mathcal{V}$ everywhere, where $\mathcal{V}'$ is not necessary integrable. Since $TM = \mathcal{V} \oplus \mathcal{V}'$, the map $\varpi_* : \mathcal{V}' \xrightarrow{\sim} \varpi^* TB$ is an isomorphism, and we can take $e_0, e_1 \in \Gamma(\mathcal{V}')$ so that $\varpi_*(e_i) = \partial_{y^i}$ for $i = 0, 1$. If we put $\alpha_i = e_i - \partial_{x^i}$, then $\alpha_i \in \Gamma(\mathcal{V})$, so (1) holds.

Now ϕ_0 and ϕ_1 are uniquely determined so that (2) holds. Actually, if we put $\phi_0 = a\partial_{x^2} + b\partial_{x^3}$, then we have

$$\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} g(\partial_{x^1}, \partial_{x^2}) & g(\partial_{x^1}, \partial_{x^3}) \\ g(\partial_{x^0}, \partial_{x^2}) & g(\partial_{x^0}, \partial_{x^3}) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

from $g(e_1, \phi_0) = -1$ and $g(e_0, \phi_0) = 0$. If the 2×2 matrix in the right hand side is not invertible, then there is a pair of real numbers $(p, q) \neq (0, 0)$ such that $g(\partial_{x^0}, p\partial_{x^2} + q\partial_{x^3}) = g(\partial_{x^1}, p\partial_{x^2} + q\partial_{x^3}) = 0$, and then $g(\partial_{x^\mu}, p\partial_{x^2} + q\partial_{x^3}) = 0$ for $\mu = 0, 1, 2, 3$. This contracts to the non-degeneracy of g, so the matrix is invertible, and (a, b) is determined uniquely. ϕ_1 is determined uniquely in a similar way. □

We denote as ω the connection form of the Levi-Civita connection with respect to the null tetrad $(e_0, e_1, \phi_0, \phi_1)$. Then ω is a $\mathfrak{o}(2, 2)$-valued 1-form, and we denote the elements in the same way as in (10).

Lemma 11. *The following equations hold:*

$$\begin{matrix} e(\phi_0) = e(\phi_1) = 0, & \\ e(e_0) = a(\phi_0) = c(\phi_1), & a(\phi_1) = c(\phi_0) = 0, \\ e(e_1) = b(\phi_0) = d(\phi_1), & b(\phi_1) = d(\phi_0) = 0, \end{matrix} \tag{16}$$

$$\begin{aligned} &[\phi_0, \phi_1] = (b(\phi_0) + d(\phi_1))\phi_0 - (a(\phi_0) + c(\phi_1))\phi_1, \\ &[e_0, \phi_0] = -(d(e_0) + f(\phi_0))\phi_0 + c(e_0)\phi_1, \\ &[e_0, \phi_1] = (b(e_0) - f(\phi_1))\phi_0 - a(e_0)\phi_1, \\ &[e_1, \phi_0] = -d(e_1)\phi_0 + (c(e_1) - f(\phi_0))\phi_1, \\ &[e_1, \phi_1] = b(e_1)\phi_0 - (a(e_1) + f(\phi_1))\phi_1. \end{aligned} \tag{17}$$

Proof. Since the Levi-Civita connection ∇ is torsion-free, we have

$$\begin{aligned} [\phi_0, \phi_1] &= \nabla_{\phi_0}\phi_1 - \nabla_{\phi_1}\phi_0 \\ &= \{e(\phi_0)e_1 + b(\phi_0)\phi_0 - a(\phi_0)\phi_1\} - \{e(\phi_1)e_0 - d(\phi_1)\phi_0 + c(\phi_1)\phi_1\}. \end{aligned}$$

Since $\mathcal{V} = \langle \phi_0, \phi_1 \rangle$ is integrable, we have $[\phi_0, \phi_1] \in \mathcal{V}$. Then we obtain

$$e(\phi_1) = e(\phi_0) = 0 \tag{18}$$

and the equation for $[\phi_0, \phi_1]$ in (17). By a calculation similar to that for $[e_i, \phi_j] \in \mathcal{V}$, we can check all the equations. □

In the rest of this section, we assume an additional condition: the neutral metric g is self-dual. Then $\mathfrak{m}_1$ and $\mathfrak{m}_2$ defined in (15) satisfy the following properties.

Lemma 12. $Q_2(\zeta) = 0$ *and* $(\phi_0 + \zeta\phi_1)Q_1(\zeta) = 0$*.*

Proof. Since E is integrable, $[\mathfrak{m}_1, \mathfrak{m}_2] \subset \langle \mathfrak{m}_1, \mathfrak{m}_2 \rangle$. Now we have

$$\begin{aligned}[\mathfrak{m}_1, \mathfrak{m}_2] &= [e_0, \phi_0] + \zeta^2[e_1, \phi_1] + \zeta([e_0, \phi_1] + [e_1, \phi_0]) + (e_0 + \zeta e_1)Q_2(\zeta)\partial_\zeta - Q_2(\zeta)e_1 \\ &\quad + Q_1(\zeta)\phi_1 - (\phi_0 + \zeta\phi_1)Q_2(\zeta)\partial_\zeta + (Q_1(\zeta)Q_2'(\zeta) - Q_2(\zeta)Q_1'(\zeta))\partial_\zeta.\end{aligned}$$

Since $[e_0, \phi_0] \in \mathcal{V}$ and so on, we can write $[\mathfrak{m}_1, \mathfrak{m}_2] = \beta(\zeta)\mathfrak{m}_2$ by using some function $\beta(\zeta)$. At the same time, we obtain the required equations. $\square$

Lemma 13. *The following equations hold:*

$$e = 0, \qquad a(\phi_i) = b(\phi_i) = c(\phi_i) = d(\phi_i) = 0, \quad b = c. \tag{19}$$

In particular we obtain $[\phi_0, \phi_1] = 0$ *from* (17).

Proof. From $Q_2(\zeta) = 0$, we have

$$\begin{aligned}&b(\phi_0) = 0, \qquad (d - a)(\phi_0) + b(\phi_1) = 0,\\ &c(\phi_1) = 0, \qquad (a - d)(\phi_1) + c(\phi_0) = 0.\end{aligned}$$

Then the first and the second equations in (19) follow from these equations and (16).

Now let $Q_1(\zeta) = q_0 + q_1\zeta + q_2\zeta^2 + q_3\zeta^3$, i.e.

$$\begin{aligned}&q_0 = -b(e_0), \qquad q_1 = -(d - a)(e_0) - b(e_1),\\ &q_3 = c(e_1), \qquad q_2 = (a - d)(e_1) + c(e_0).\end{aligned} \tag{20}$$

We can write $[\mathfrak{m}_1, \mathfrak{m}_2] = \beta(\zeta)\mathfrak{m}_2$ from the proof of Lemma 12, and we can put $\beta(\zeta) = \beta_0 + \beta_1\zeta + \beta_2\zeta^2$ from the relation of the degree; then from a direct calculation, we obtain

$$\begin{aligned}&[e_0, \phi_0] = \beta_0\phi_0 - q_0\,\phi_1,\\ &[e_0, \phi_1] + [e_1, \phi_0] = \beta_1\phi_0 + (\beta_0 - q_1)\phi_1,\\ &[e_1, \phi_1] = \beta_2\phi_0 + (\beta_1 - q_2)\phi_1,\\ &0 = (\beta_2 - q_3)\phi_1.\end{aligned} \tag{21}$$

Comparing with (17), and using (20), we have $b(e_i) = c(e_i)$. Since we already have $b(\phi_i) = c(\phi_i)$, so we obtain $b = c$. $\square$

Lemma 14. *The following equations hold:*

$$\begin{aligned}&\phi_0 b(e_0) = \phi_1 b(e_1) = 0,\\ &\phi_0(a - d)(e_0) = \phi_1(d - a)(e_1) = \phi_0 b(e_1) + \phi_1 b(e_0),\\ &\phi_0(a - d)(e_1) = \phi_1(d - a)(e_0).\end{aligned} \tag{22}$$

Proof. Directly deduced from (20) and $(\phi_0 + \zeta\phi_1)Q_1(\zeta) = 0$. $\square$

Proposition 15. *Let g be a neutral self-dual metric on a four-dimensional manifold M and $\varpi : M \to B$ be an α-surface foliation. Then there is a local coordinate (x^0, x^1, x^2, x^3) on M so that $\ker \varpi_* = \langle \partial_{x^2}, \partial_{x^3} \rangle$ and that the metric tensor for g can be written in the form*

$$g = (g_{ij}) = \begin{pmatrix} p & r & 0 & 1\\ r & q & -1 & 0\\ 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}. \tag{23}$$

Moreover, p, q and r satisfy the following equations:

$$\begin{cases}\partial_2^2 p = \partial_3^2 q = 0,\\ \partial_3^2 p + \partial_2^2 q = 0,\\ \partial_2^2 r + \partial_2\partial_3 p = \partial_3^2 r + \partial_2\partial_3 q = 0,\end{cases} \tag{24}$$

where $\partial_\mu = \partial_{x^\mu}$. Conversely, for any functions p, q and r satisfying (24), the neutral metric defined by (23) is self-dual and has a natural α-surface foliation.

Proof. For given (M, g) and ϖ, we can take a coordinate (x^0, x^1, x^2, x^3) and a null tetrad $(e_0, e_1, \phi_0, \phi_1)$ on M as in Proposition 10. Since we have $[\phi_0, \phi_1] = 0$ from Lemma 13, we can change the coordinates x^2, x^3 to w^2, w^3 so that $\phi_0 = \partial_{w^2}, \phi_1 = \partial_{w^3}$. So we can start from $\phi_0 = \partial_{x^2}, \phi_1 = \partial_{x^3}$. Then the metric tensor is written in the form (23), since we have

$$
\begin{aligned}
&g(\partial_0, \partial_2) = g(e_0 - \alpha_0, \phi_0) = 0,\\
&g(\partial_1, \partial_2) = g(e_1 - \alpha_1, \phi_0) = -1,
\end{aligned}
$$

and so on.

Now we check that the metric in the form (23) is self-dual if and only if (24) holds. We take a frame on TM of the form

$$
\begin{cases} e_0 = \partial_0 + \dfrac{1}{2}(r\partial_2 - p\partial_3),\\ e_1 = \partial_1 + \dfrac{1}{2}(q\partial_2 - r\partial_3), \end{cases} \qquad \begin{cases} \phi_0 = \partial_2,\\ \phi_1 = \partial_3. \end{cases} \tag{25}
$$

Then $(e_0, e_1, \phi_0, \phi_1)$ is a null tetrad satisfying the conditions of Proposition 10. Calculating $[e_i, \phi_j]$ and so on, and comparing with (17), we have

$$
\begin{aligned}
&2a(e_0) = -\partial_3 p, \qquad 2b(e_0) = \partial_2 p, \qquad 2d(e_0) = \partial_3 q + 2\partial_2 r\\
&2a(e_1) = -\partial_2 p - 2\partial_3 r, \qquad 2b(e_1) = -\partial_3 q, \qquad 2d(e_1) = \partial_2 q.
\end{aligned} \tag{26}
$$

We obtain (24) by evaluating these equations using (22). $\square$

Remark 16. The form of the metric (23) coincides with the Walker canonical form (Theorem 1 of [14]; see also [9, 10]), and in the special case with the ASD null Kähler canonical form (Theorem 3.2 of [3]).

5. Basic foliation

Let (M, g) be a space–time oriented 4-manifold with a neutral metric, and let $\varpi : M \to B$ be an α-surface foliation.

Definition 17. We define ϖ as basic if and only if the curvature Ω^+ of the spin connection ∇^+ on S^+ defined by (7) is basic, i.e. $i(v)\Omega^+ = 0$ for every vertical vector $v \in \ker \varpi_*$.

We use the same local descriptions as in Section 4. Then the following lemma is proved by a direct calculation.

Lemma 18. *If g is self-dual, then ϖ is basic if and only if*

$$
\phi_i b(e_j) = \phi_i (a - d)(e_j) = 0 \quad \text{for } i, j = 0, 1. \tag{27}
$$

Moreover (27) is equivalent to the equations $\phi_i q_n = 0$ for $i = 0, 1$ and $n = 0, 1, 2, 3$, where $Q_1(\zeta) = q_0 + q_1\zeta + q_2\zeta^2 + q_3\zeta^3$.

Proposition 19. *Suppose that ϖ is basic, and that g is self-dual; then ϖ is also basic for the conformal deformation $\tilde{g} = \varphi g$, where φ is a non-vanishing function on M.*

Proof. Let $(e_0, e_1, \phi_0, \phi_1)$ be a null tetrad on (M, g) which satisfies the conditions in Proposition 10; then $(\tilde{e}_0, \tilde{e}_1, \tilde{\phi}_0, \tilde{\phi}_1) = (e_0, e_1, \varphi^{-1}\phi_0, \varphi^{-1}\phi_1)$ is a null tetrad on $(M, \tilde{g})$ and satisfies the same conditions. Let $\mathfrak{m}_1, \mathfrak{m}_2$ be the frame of the twistor distribution defined by (15) with respect to g. In the same way, we define $\tilde{\mathfrak{m}}_1, \tilde{\mathfrak{m}}_2$ with respect to $\tilde{g}$, and we define

$$
\tilde{\mathfrak{m}}_1 = \tilde{e}_0 + \zeta\tilde{e}_1 + \tilde{Q}_1(\zeta)\partial_\zeta, \qquad \tilde{Q}_1(\zeta) = \tilde{q}_0 + \tilde{q}_1\zeta + \tilde{q}_2\zeta^2 + \tilde{q}_3\zeta^3,
$$

and so on. Since $Q_2(\zeta) = \tilde{Q}_2(\zeta) = 0$ in this case, we have $\tilde{\mathfrak{m}}_2 = \varphi^{-1}\mathfrak{m}_2$. Now $\langle \mathfrak{m}_1, \mathfrak{m}_2\rangle = \langle \tilde{\mathfrak{m}}_1, \tilde{\mathfrak{m}}_2\rangle = E_{\mathbb{R}}$ by Theorem 7, so we obtain $\tilde{\mathfrak{m}}_1 = \mathfrak{m}_1$ and $\tilde{q}_n = q_n$. Then, for every $i = 0, 1$ and $n = 0, 1, 2, 3$, we have $\tilde{\phi}_i\tilde{q}_n = \varphi^{-1}\phi_i q_n = 0$ by Lemma 18. Hence ϖ is also basic for $\tilde{g}$. □

Proposition 20 (*Cf.* [2]). *Let (M, g) be a 4-manifold with a neutral self-dual metric and $\varpi : M \to B$ be an α-surface foliation. If ϖ is basic, then there is a unique projective structure $[\nabla]$ which satisfies the following condition:*

- *the image of each β-surface by ϖ is a geodesic on B.*

Conversely, if the above condition holds for some projective structure on B, then ϖ is basic.

Proof. We take coordinate neighborhoods $U \subset M$ and $V = \varpi(U) \subset B$ so that the coordinates are written in the manner of Section 4. Let $(e_0, e_1, \phi_0, \phi_1)$ be the null tetrad given by Proposition 10. Using the trivialization of $\mathcal{Z}_{\mathbb{R}}$, $(x,\zeta) \in U \times \mathbb{RP}^1 \cong \mathcal{Z}_{\mathbb{R}}|_U$ corresponds to the β-surface $\beta(\zeta) = \langle e_0 + \zeta e_1, \phi_0 + \zeta\phi_1\rangle_x$. Then we have $\varpi_*(\beta(\zeta)) = \langle \partial_{y^0} + \zeta\partial_{y^1}\rangle_{\varpi(x)}$, and this is a line in $T_{\varpi(x)}B$ that corresponds to the point $(\varpi(x), \zeta) \in V \times \mathbb{RP}^1 \simeq \mathcal{W}_{\mathbb{R}}$. In this way, we obtain a map $\Pi_{\mathbb{R}} : \mathcal{Z}_{\mathbb{R}} \to \mathcal{W}_{\mathbb{R}}$ which extends holomorphically to the map $\Pi : \mathcal{Z}_+ \to \mathcal{W}_+$.

Using the above coordinates, we have

$$\begin{aligned} &\Pi_*(\mathfrak{m}_1) = \partial_0 + \zeta\partial_1 + Q_1(\zeta)\partial_\zeta, \qquad Q_1(\zeta) = q_0 + q_1\zeta + q_2\zeta^2 + q_3\zeta^3,\\ &\Pi_*(\mathfrak{m}_2) = 0. \end{aligned} \tag{28}$$

If there is a projective structure $[\nabla]$ on B satisfying the condition in the statement, then $\Pi_*(E) = L$, i.e. $\langle \mathfrak{n}\rangle = \langle \Pi_*(\mathfrak{m}_1)\rangle$. This equation holds only if $\phi_i q_n = 0$, so ϖ is basic by Lemma 18.

Conversely, if ϖ is basic, then $\Pi_*(E)$ defines a complex distribution on $\mathcal{W}_+$. Then we can define a torsion-free connection on B so that $\mathfrak{n} = \Pi_*(\mathfrak{m}_1)$. Actually, one of the examples of such a connection is given as follows. Now b and $a - d$ define a 1-form on V from (27), so we can define a connection whose connection form (ω^i_j) is given by $\omega^0_1 = \omega^1_0 = b$, and

$$\begin{aligned} &\omega^0_0(\partial_{y^0}) = (a-d)(\partial_{y^0}) + b(\partial_{y^1}), \qquad \omega^0_0(\partial_{y^1}) = b(\partial_{y^0}),\\ &\omega^1_1(\partial_{y^0}) = b(\partial_{y^1}), \qquad \omega^1_1(\partial_{y^1}) = (d-a)(\partial_{y^1}) + b(\partial_{y^0}). \end{aligned}$$

Then this connection is torsion-free and the equation $\mathfrak{n} = \Pi_*(\mathfrak{m}_1)$ holds on $V = \varpi(U)$. This means that the condition in the statement holds. Since the projective structure is exactly classified by the geodesics, such a projective structure $[\nabla]$ is uniquely defined. □

Example 21. Let (x^0, x^1, x^2, x^3) be a coordinate on $\mathbb{R}^4$, and consider a metric g on $\mathbb{R}^4$ whose metric tensor $g_{\mu\nu} = g(\partial_{x^\mu}, \partial_{x^\nu})$ is given by

$$g = (g_{ij}) = \begin{pmatrix} p & r & 0 & 1\\ r & p & -1 & 0\\ 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \text{where } \begin{cases} p = -2x^2x^3,\\ r = (x^2)^2 + (x^3)^2. \end{cases}$$

Then g is neutral and self-dual; however the α-surface foliation defined from the integrable distribution $\mathcal{V} = \langle \partial_{x^2}, \partial_{x^3}\rangle$ is not *basic*. Actually, if we take a null tetrad in the form of (25), then we have

$$\phi_0 b(e_1) = \phi_1 b(e_0) = 2 \neq 0, \tag{29}$$

so (27) does not hold. Note that the above metric has in fact constant curvature.

6. Global structure: Main theorem and the mini-twistor version

In this section and Section 7, we treat the global structure. From now on, we write simply "β-surface" for the maximal β-surface. The following properties are proved by LeBrun and Mason in [8].

Proposition 22. *Let $(M, [g])$ be a space–time oriented self-dual Zollfrei conformal structure; then*

(1) *any two β-surfaces intersect at exactly two points,*
(2) *every β-surface is a totally geodesic embedded S^2,*

(3) *for every β-surface β, the restriction of the Levi-Civita connection of g to β defines a Zoll projective structure which depends only on the conformal class $[g]$; moreover this is isomorphic to the standard Zoll projective structure on S^2.*

Suppose there is a closed α-surface on M; then it satisfies the following lemma.

Lemma 23. *Let $(M, [g])$ be a space–time oriented self-dual Zollfrei conformal structure, and let α be a closed α-surface on M, then*

(1) *α is a totally geodesic embedded S^2,*
(2) *the restriction of the Levi-Civita connection of g to α defines a Zoll projective structure which depends only on the conformal class $[g]$,*
(3) *for every β-surface β, the intersection of α and β is either empty or S^1 which is a geodesic on both α and β for the induced projective structure.*

Proof. In the same way as for Proposition 22, we can check that α is totally geodesic and that $[g]$ induces a projective structure on α. Then (1) and (2) follow from the Zollfrei condition. Let β be a β-surface; then $\alpha \cap \beta$ is totally geodesic in M. This is either empty or a one-dimensional manifold, since any α-plane and any β-plane intersect in a one-dimensional subspace at a point. So this is a closed geodesic on M. Since α and β are totally geodesic, (3) holds. $\square$

We now study self-dual Zollfrei conformal structures with α-surface foliation.

Definition 24. Let $(M, [g], S_\infty, \mathcal{F})$ be the quartet of the space–time oriented self-dual Zollfrei conformal structure $(M, [g])$, a β-surface S_∞ and a family $\mathcal{F}$ of closed α-surfaces which satisfies the following properties: (i) every α-surface $\alpha \in \mathcal{F}$ has non-empty intersection with S_∞, (ii) $\mathcal{F}$ defines a smooth foliation on $M \setminus S_\infty$. Two such quartets are said to be equivalent if and only if there is a conformal isomorphism between them which preserves S_∞ and $\mathcal{F}$. We define $\mathcal{M}$ to be the set of equivalence classes of such quartets.

Definition 25. We define $\bar{\mathcal{M}}$ to be the set of conformal equivalence classes of space–time oriented self-dual Zollfrei conformal structures $(M, [g])$. Then we have a natural forgetting map $\mathcal{M} \to \bar{\mathcal{M}}$.

If there are no confusions, we abuse the notation of a quartet $(M, [g], S_\infty, \mathcal{F})$ for its equivalence class, and similarly for a pair $(M, [g])$.

Lemma 26. *Let $(M, [g], S_\infty, \mathcal{F})$ be an element of $\mathcal{M}$, and let β be a β-surface different from S_∞; then $\beta \cap S_\infty$ is the set of antipodal points of β with respect to the induced standard Zoll projective structure on β. Moreover, for $\alpha \in \mathcal{F}$, $\alpha \cap \beta$ contains $\beta \cap S_\infty$ if $\alpha \cap \beta$ is not empty.*

Proof. If we take a point in $\beta \setminus S_\infty$, then there is a unique $\alpha_1 \in \mathcal{F}$ which contains this point. If we take the other point on $\beta \setminus (S_\infty \cup \alpha_1)$, then there is a unique $\alpha_2 \in \mathcal{F}$ again which contains this point. Then $\alpha_1 \cap \beta$ and $\alpha_2 \cap \beta$ are different geodesics on β, so $\alpha_1 \cap \alpha_2 \cap \beta$ equals the set of antipodal points of β. These points belong to both α_1 and α_2, so they must belong to S_∞. On the other hand, $\beta \cap S_\infty$ is just two points, so $\beta \cap S_\infty = \alpha_1 \cap \alpha_2 \cap \beta$. Hence $\beta \cap S_\infty$ is the set of antipodal points of β. The latter statement is now obvious. $\square$

Lemma 27. *For $(M, [g], S_\infty, \mathcal{F}) \in M$, each α-surface in $\mathcal{F}$ one-to-one corresponds with a closed geodesic of S_∞.*

Proof. Each α-surface $\alpha \in \mathcal{F}$ determines a closed geodesic $\alpha \cap S_\infty$ on S_∞. We prove that this correspondence is bijective. The injectivity follows at once since the α-surface is totally geodesic. So we check the surjectivity. It is enough to show that, for each $x \in S_\infty$ and each one-dimensional subspace $l \subset T_x S_\infty$, there is an α-surface $\alpha \in \mathcal{F}$ such that $T_x\alpha \cap T_x S_\infty = l$. There is a unique α-plane $H \subset T_x M$ which contains l, and we can take a one-dimensional subspace $l' \subset H$ different from l. Let c be a closed null geodesic of M which is tangent to l' at x. We can take $y \in c \setminus S_\infty$ since l' is not tangent to S_∞. Then there is a unique α-surface $\alpha \in \mathcal{F}$ containing y, and there is a unique β-surface β with $c \subset \beta$. Since $y \in \alpha \cap \beta$, $\alpha \cap \beta$ is non-empty and is a closed geodesic on β. Since $\alpha \cap \beta$ contains $\beta \cap S_\infty$ and y by Lemma 26, this is equal to c. Then we have $x \in \alpha$ and $T_x\alpha = H$, so we have $T_x\alpha \cap T_x S_\infty = l$ as required. $\square$

Let $\tilde{\mathcal{G}}(S_\infty)$ be the set of oriented closed geodesics on S_∞. $\tilde{\mathcal{G}}(S_\infty)$ has natural smooth structure since the induced projective structure on S_∞ is standard. $\tilde{\mathcal{G}}(S_\infty)$ is diffeomorphic to S^2, and has natural Zoll projective structure induced from S_∞ so that a geodesic on $\tilde{\mathcal{G}}(S_\infty)$ corresponds to the set of oriented geodesics on S_∞ containing one fixed point.

Proposition 28. *Let $(M, [g], S_\infty, \mathcal{F})$ be an element of $\mathcal{M}$; then there is a natural identification between $\tilde{\mathcal{G}}(S_\infty)$ and the leaf space B of the foliation on $M \setminus S_\infty$ defined by $\mathcal{F}$.*

Proof. For every $\alpha \in \mathcal{F}$, α is diffeomorphic to S^2, and $S_\infty \cap \alpha = S^1$, so $(M \setminus S_\infty) \cap \alpha$ is disjoint union of a pair of disks. Hence $M \setminus S_\infty$ is foliated by such disks. Each $\alpha \in \mathcal{F}$ has natural orientation defined from the space–time orientation on M, so each disk of the foliation is oriented. Then the natural orientation is induced on the boundary of each disk. In this way, the leaf space B naturally corresponds to $\tilde{\mathcal{G}}(S_\infty)$. □

Proposition 29. *For $(M, [g], S_\infty, \mathcal{F}) \in \mathcal{M}$, the α-surface foliation on $M \setminus S_\infty$ induced from $\mathcal{F}$ is basic, and the projective structure induced on the leaf space B is isomorphic to the standard Zoll projective structure.*

Proof. We already know that $B = \tilde{\mathcal{G}}(S_\infty)$ has the standard Zoll projective structure induced from S_∞. We now check that this projective structure equals the one induced from the α-surface foliation. Then this α-surface foliation is automatically basic from Proposition 20.

Let β be any β-surface different from S_∞. It is enough to check that the set of all the leaves intersecting with β corresponds to some closed geodesic on B with respect to the above Zoll projective structure. From Lemma 26, an α-surface $\alpha \in \mathcal{F}$ intersects β if and only if $\alpha \cap S_\infty$ contains the antipodal points $\beta \cap S_\infty$. Hence the set of α-surfaces in $\mathcal{F}$ intersecting β corresponds to the set of closed geodesics on S_∞ containing $\beta \cap S_\infty$ under the correspondence of Lemma 27. Such a set is a closed geodesic on $\tilde{\mathcal{G}}(S_\infty)$. □

Let $\mathbb{RP}^n \subset \mathbb{CP}^n$ be the standard real submanifold.

Definition 30. Let (ι, ζ_0) be the pair of a totally real embedding $\iota : \mathbb{RP}^3 \to \mathbb{CP}^3$ and a point $\zeta_0 \in P = \iota(\mathbb{RP}^3)$ which satisfy:

- $\pi(P \setminus \{\zeta_0\}) = \mathbb{RP}^2$ for the projection $\pi : \mathbb{CP}^3 \setminus \{\zeta_0\} \to \mathbb{CP}^2$, where $\mathbb{CP}^2$ is the space of complex lines in $\mathbb{CP}^3$ through ζ_0, and this equation means that the image $\pi(P \setminus \{\zeta_0\})$ is mapped to the standard $\mathbb{RP}^2$ by some holomorphic automorphism on $\mathbb{CP}^2$,
- let $\mathbb{CP}^1_\xi = \pi^{-1}(\xi) \cap \{\zeta_0\}$ and $P_\xi = \mathbb{CP}^1_\xi \cap P$ for each $\xi \in \pi(P \setminus \{\zeta_0\})$; then $(\mathbb{CP}^1_\xi, P_\xi)$ is biholomorphic to $(\mathbb{CP}^1, \mathbb{RP}^1)$, i.e. there is a biholomorphic map $\mathbb{CP}^1_\xi \to \mathbb{CP}^1$ which maps P_ξ to $\mathbb{RP}^1$.

Two such pairs (ι, ζ_0) and (ι', ζ_0') are said to be equivalent if and only if there is a holomorphic automorphism φ on $\mathbb{CP}^3$ which satisfies $\iota' = \varphi \circ \iota$ and $\zeta_0' = \varphi(\zeta_0)$. We define $\mathcal{T}$ to be the set of equivalence classes of such pairs.

Definition 31. We define $\bar{\mathcal{T}}$ to be the set of equivalence classes of totally real embeddings $\iota : \mathbb{RP}^3 \to \mathbb{CP}^3$. Then we have a natural forgetting map $\mathcal{T} \to \bar{\mathcal{T}}$.

We abuse the notation (ι, ζ_0) or ι for their equivalence classes. Our main theorem is the following. We define $f_{\mathcal{M}} : \mathcal{M} \to \bar{\mathcal{M}}$ and $f_{\mathcal{T}} : \mathcal{T} \to \bar{\mathcal{T}}$ as the forgetting maps.

Theorem 32. *Let $U \subset \bar{\mathcal{M}}$ and $V \subset \bar{\mathcal{T}}$ be subsets containing the standard elements on which the one-to-one correspondence in the sense of Theorem 9 holds. Then there is a one-to-one correspondence between $f_{\mathcal{M}}^{-1}(U)$ and $f_{\mathcal{T}}^{-1}(V)$ which satisfies the following properties: if $(M, [g], S_\infty, \mathcal{F})$ corresponds to (ι, ζ_0), then*

(1) *$(M, [g])$ corresponds to ι in the sense of Theorem 9, i.e. this correspondence covers the correspondence between U and V,*
(2) *the standard double fibration $B \leftarrow \mathcal{W}_+ \to \mathbb{CP}^2$ is induced by using the maps $\varpi : M \setminus S_\infty \to B$ and $\pi : \mathbb{CP}^3 \setminus \{\zeta_0\} \to \mathbb{CP}^2$, where ϖ is the α-surface foliation defined from $\mathcal{F}$ and π is the projection from ζ_0.*

Before we start to prove Theorem 32, we argue about a mini-twistor version in the rest of this section. The situation is described in the diagram (30).

Definition 33. Let $(S^2, [\nabla], C)$ be the triple of an oriented Zoll projective structure $(S^2, [\nabla])$ and a closed geodesic C on S^2 which satisfies: (i) there is a smooth involution σ on C, and (ii) for every $x \in C$ and every closed geodesic c through x, c passes through $\sigma(x)$. We call $\sigma(x)$ the antipodal point of x and write $\bar{x}$ for $\sigma(x)$. Two such triples are said to be equivalent if and only if there is an automorphism on S^2 which preserves $[\nabla]$, C, and the involution. We define $\mathcal{M}_0$ to be the set of equivalence classes of such triples.

Definition 34. We define $\bar{\mathcal{M}}_0$ to be the set of equivalence classes of oriented Zoll projective structures $(S^2, [\nabla])$. Then we have a forgetting map $\mathcal{M}_0 \to \bar{\mathcal{M}}_0$.

Definition 35. Let (ι, ζ_0) be the pair of a totally real embedding $\iota : \mathbb{RP}^2 \to \mathbb{CP}^2$ and a point $\zeta_0 \in N = \iota(\mathbb{RP}^2)$ which satisfy:

- $\pi(N \setminus \{\zeta_0\}) = \mathbb{RP}^1$ where $\pi : \mathbb{CP}^2 \setminus \{\zeta_0\} \to \mathbb{CP}^1$ is the projection,
- let $\mathbb{CP}^1_\xi = \pi^{-1}(\xi) \cup \{\zeta_0\}$ and $N_\xi = \mathbb{CP}^1_\xi \cap N$; then $(\mathbb{CP}^1_\xi, P_\xi)$ is biholomorphic to $(\mathbb{CP}^1, \mathbb{RP}^1)$.

We define $\mathcal{T}_0$ to be the set of equivalence classes of such pairs, where the equivalence is defined in the same manner as in Definition 30.

Definition 36. We define $\bar{\mathcal{T}}_0$ to be the set of equivalence classes of the totally real embeddings $\iota : \mathbb{RP}^2 \to \mathbb{CP}^2$. Then we have a forgetting map $\mathcal{T}_0 \to \bar{\mathcal{T}}_0$.

We abuse the notation $(S^2, [\nabla], C)$ for its equivalence class and so on, and we define $f_{\mathcal{M}_0} : \mathcal{M}_0 \to \bar{\mathcal{M}}_0$ and $f_{\mathcal{T}_0} : \mathcal{T}_0 \to \bar{\mathcal{T}}_0$ as the forgetting maps.

Theorem 37. *Let $U_0 \subset \bar{\mathcal{M}}_0$ and $V_0 \subset \bar{\mathcal{T}}_0$ be subsets containing the standard elements on which the one-to-one correspondence in the sense of Theorem 3 holds. Then there is a one-to-one correspondence between $f^{-1}_{\mathcal{M}_0}(U_0)$ and $f^{-1}_{\mathcal{T}_0}(V_0)$ which covers the correspondence between U_0 and V_0.*

Proof. We start from an element $(S^2, [\nabla], C) \in \mathcal{M}_0$. If $(S^2, [\nabla]) \in U_0$, then we have a double fibration $S^2 \xleftarrow{p_1} (\mathcal{W}_+, \mathcal{W}_\mathbb{R}) \xrightarrow{q_1} (\mathbb{CP}^2, N)$ from Theorem 3, where N is the image of the totally real embedding $\iota : \mathbb{RP}^2 \to \mathbb{CP}^2$. We define $\zeta_0 \in N$ to be the point corresponding to C, i.e. ζ_0 is the point such that $C = p_1 \circ q_1^{-1}(\zeta_0)$. Let $x \in C$ be any point, and $\bar{x}$ be its antipodal point, and D_x, $D_{\bar{x}}$ be the holomorphic disks on $(\mathbb{CP}^2, \mathbb{N})$, i.e. $D_x = q_1 \circ p_1^{-1}(x)$ and so on. Notice that $\zeta_0 \in D_x$ and $\zeta_0 \in D_{\bar{x}}$. Since each point on $\partial D_x \subset N$ corresponds to some closed geodesic on S^2 containing x, and since such geodesics also contain $\bar{x}$ from the definition, we have $\partial D_x = \partial D_{\bar{x}}$. Hence $l_x = D_x \cup D_{\bar{x}}$ defines a rational curve on $\mathbb{CP}^2$, and this is proved to be a complex line. Actually, let $y \in C$ be a point different from x and $\bar{x}$, and l_y be a rational curve defined in the same way as above. Then $l_x \cap l_y = \{\zeta_0\}$; moreover ∂D_x and ∂D_y intersect transversely in N, so l_x and l_y intersect only on ζ_0 transversely. Hence l_x must be a complex line.

Let $\pi : \mathbb{CP}^2 \setminus \{\zeta_0\} \to \mathbb{CP}^1$ be a natural projection. From the above argument, we see that π maps $l_x \setminus \{\zeta_0\}$ to a point. $N \setminus \{\zeta_0\}$ is foliated by lines in the form of $\partial D_x \setminus \{\zeta_0\}$, and such a line one-to-one corresponds to a pair of antipodal points $\{x, \bar{x}\}$ in C. Since $\pi(N \setminus \{\zeta_0\})$ is the quotient space of such a line foliation, it is diffeomorphic to $C/\mathbb{Z}_2 \simeq \mathbb{RP}^1$. Since N is a totally real embedded $\mathbb{RP}^2$, it follows that $\pi(N \setminus \{\zeta_0\})$ is also a totally real submanifold in $\mathbb{CP}^1$. Hence $(\mathbb{CP}^1, \pi(N \setminus \{\zeta_0\}))$ is biholomorphic to $(\mathbb{CP}^1, \mathbb{RP}^1)$, and (ι, ζ_0) defines an element of $\mathcal{T}$.

Next we start from $(\iota, \zeta_0) \in \mathcal{T}$. If $\iota \in V_0$, then we have a double fibration $S^2 \xleftarrow{p_1} (\mathcal{W}_+, \mathcal{W}_\mathbb{R}) \xrightarrow{q_1} (\mathbb{CP}^2, N)$. We define $C = p_1 \circ q_1^{-1}(\zeta_0)$, and we prove that there is a natural involution σ on C.

$(\mathbb{CP}^1_\xi, N_\xi)$ consists of two holomorphic disks D_1 and D_2 with $\partial D_1 = \partial D_2 = N_\xi$ for every $\xi \in \mathbb{RP}^1$. Since D_1 and D_2 define generators of $H_2(\mathbb{CP}^2, N)$, they correspond to some points x_1 and x_2 in S_∞ respectively. Then $x_1 \in C$ from $\zeta_0 \in \partial D_1$, and similarly $x_2 \in C$. Now all the holomorphic disks containing ζ_0 are written in the above form, so C equals to the union of such pairs of points. We define σ to be the involution on C interchanging two such points.

It is enough to show that every closed geodesic in S^2 through $x \in C$ always passes through $\bar{x} = \sigma(x)$. This is, however, obvious because each closed geodesic through $x \in C$ corresponds to some point on $\partial D_x = \partial D_{\bar{x}}$ under the double fibration, so this geodesic also passes through $\bar{x}$. □

Now we explain the following diagram as regards Theorem 37.

$$
\begin{tikzcd}
 & \mathcal{W}_+ \arrow[dl, "p_1"'] \arrow[dr, "q_1"] \arrow[d, phantom, "\cup"] & \\
S^2 \arrow[d, phantom, "\cup"] & \mathcal{W}_+^r \arrow[dl, "p_1^r"'] \arrow[d, "\Pi"] \arrow[dr, "q_1^r"] & \mathbb{CP}^2 \arrow[d, phantom, "\cup"] \\
S^2 \setminus C \arrow[d, "\varpi"] & D_+ \sqcup D_- \arrow[dl, "p_0"'] \arrow[dr, "q_0"] & \mathbb{CP}^2 \setminus \{\zeta_0\} . \arrow[d, "\pi"] \\
\{b_\pm\} & & \mathbb{CP}^1
\end{tikzcd}
\tag{30}
$$

Let D_+ be one of the two holomorphic disks of $(\mathbb{CP}^1, \mathbb{RP}^1)$ and let D_- be the other one. Let $q_0 : D_+ \sqcup D_- \to \mathbb{CP}^1$ be the natural map. Let $\{b_\pm\}$ be a set consisting of two points, and define a map p_0 by $p_0(D_\pm) = b_\pm$. We define $\mathcal{W}_+^r = p_1^{-1}(S^2 \setminus C)$, and let p_1^r and q_1^r be the restrictions of p_1 and q_1. Then Π and ϖ are naturally induced from π so that the diagram commutes. Notice that ϖ maps each connected component of $S^2 \setminus C$ to one point. We define $S^2_\pm = \varpi^{-1}(b_\pm)$.

From the proof of Theorem 37, each point $\xi \in \mathbb{RP}^1$ corresponds to a pair of antipodal points $\{x, \bar{x}\}$ of C via $\mathbb{CP}^1_\xi = D_x \cup D_{\bar{x}}$. Hence there is a natural isomorphism $i : C/\mathbb{Z}_2 \xrightarrow{\sim} \mathbb{RP}^1$. On the other hand, there is a natural map $\mu : \mathcal{W}_\mathbb{R}^r \to C/\mathbb{Z}_2$ defined in the following way. Each point of $\mathcal{W}_\mathbb{R}^r$ corresponds to a pair (x, l) of a point $x \in S^2 \setminus C$ and a closed geodesic l on S^2 containing x. Then we define $\mu(x, l) \in C/\mathbb{Z}_2$ to be the intersection $l \cap C$. We have $i \circ \mu = q_0 \circ \Pi_\mathbb{R}$ by definition, where $\Pi_\mathbb{R}$ is the restriction of Π to $\mathcal{W}_\mathbb{R}^r$.

In this way, we have checked that $\Pi : \mathcal{W}_+^r \to D_+ \sqcup D_-$ satisfies the following conditions:

(Π1) Π is smooth and $\varpi \circ p_1^r = p_0 \circ \Pi$,
(Π2) there is an isomorphism $i : C/\mathbb{Z}_2 \to \mathbb{RP}^1$ satisfying $i \circ \mu = q_0 \circ \Pi_\mathbb{R}$,
(Π3) Π is holomorphic on $\mathcal{W}_+^r \setminus \mathcal{W}_\mathbb{R}^r$.

The next lemma says that such a map Π satisfying the above conditions is determined uniquely up to isomorphism.

Lemma 38. *Let Π be the map given above, and let $\Pi' : \mathcal{W}_+^r \to D_+ \sqcup D_-$ be a map which satisfies* (Π1) *to* (Π3)*. Then there is a holomorphic automorphism T on $\mathbb{CP}^1$ fixing $D_\pm$ and satisfying $\Pi' = \tilde{T} \circ \Pi$, where $\tilde{T}$ is the automorphism of $D_+ \sqcup D_-$ induced from T.*

Proof. Let $i' : C/\mathbb{Z}_2 \to \mathbb{RP}^2$ be the map satisfying the condition (Π2) for Π', i.e. $i' \circ \mu = q_0 \circ \Pi'_\mathbb{R}$. Let $x \in S^2_+$ be any point; then we have

$$i \circ \mu_x = q_0 \circ \Pi_{\mathbb{R},x}, \qquad i' \circ \mu_x = q_0 \circ \Pi'_{\mathbb{R},x},$$

where $\Pi_{\mathbb{R},x}$, $\Pi'_{\mathbb{R},x}$ and μ_x are restrictions of $\Pi_\mathbb{R}$, $\Pi'_\mathbb{R}$ and μ on $\mathcal{W}_{+,x} = p_1^{-1}(x)$. Since μ_x is bijective, we have

$$(q_0 \circ \Pi'_{\mathbb{R},x}) \circ (q_0 \circ \Pi_{\mathbb{R},x})^{-1} = i' \circ i^{-1}. \tag{31}$$

The left hand side of (31) extends holomorphically to the interior of D_+, so $i' \circ i^{-1}$ extends to a holomorphic automorphism on D_+. In the same way, if we take $x \in S_-$, we can check that $i' \circ i^{-1}$ extends to D_- holomorphically; hence there is a holomorphic automorphism T on $\mathbb{CP}^1$ fixing $D_\pm$ and satisfying $\Pi'_x = \tilde{T} \circ \Pi_x$. Since T does not depend on $x \in S \setminus C$, this is the required automorphism. □

Corollary 39. *Suppose that a given map Π satisfies* (Π1) *to* (Π3)*; then there is a unique continuous map π which makes the diagram* (30) *commute. Such a map π is equivalent to the natural projection from ζ_0.*

Proof. The map Π satisfying the conditions (Π1) to (Π3) is essentially unique, and this is the one defined from Theorem 37. So it follows that the natural projection π is the unique map which makes the diagram (30) commute. □

7. Proof of the main theorem

First we prove the following proposition.

Proposition 40. *Let $(M, [g], S_\infty, \mathcal{F}) \in \mathcal{M}$ be an element which is contained in $f_{\mathcal{M}}^{-1}(U)$ in the terminology of Theorem 32. Then there is a unique element of $\mathcal{T}$ which satisfies the conditions in Theorem 32.*

Let $(M, [g], S_\infty, \mathcal{F}) \in f_{\mathcal{M}}^{-1}(U)$ be an element; then we have a totally real embedding $\iota : \mathbb{RP}^3 \to \mathbb{CP}^3$ corresponding to $(M, [g])$ in the sense of Theorem 9. Let $P = \iota(\mathbb{RP}^3)$; then each point on P corresponds to some β-surface, so we can define $\zeta_0 \in P$ as the point corresponding to S_∞.

Let $\varpi : (M \setminus S_\infty) \to B$ be the basic α-surface foliation induced from $\mathcal{F}$. We have the standard Zoll projective structure on B by Proposition 28. Then we have the following diagram:

$$\begin{tikzcd}
 & (\mathcal{Z}_+, \mathcal{Z}_\mathbb{R}) \arrow[dl, "p_2"'] \arrow[dr, "q_2"] \arrow[d, phantom, "\cup"] & & & \\
M \arrow[d, phantom, "\cup"] & (\mathcal{Z}^r_+, \mathcal{Z}^r_\mathbb{R}) \arrow[dl, "p^r_2"'] \arrow[d, dotted, "\Pi"] \arrow[dr, "q^r_2"] & (\mathbb{CP}^3, P) \arrow[d, phantom, "\cup"] & & \\
M \setminus S_\infty \arrow[d, "\varpi"] & (\mathcal{W}_+, \mathcal{W}_\mathbb{R}) \arrow[dl, "p_1"'] \arrow[dr, "q_1"] & (q_2(\mathcal{Z}^r_+), q_{2,\mathbb{R}}(\mathcal{Z}^r_\mathbb{R})) \arrow[d, dotted, "\pi'"] & \subset & (\mathbb{CP}^3 \setminus \{\zeta_0\}, P \setminus \{\zeta_0\}) \arrow[dll, dotted, "\pi"] \\
B & & (\mathbb{CP}^2, \mathbb{RP}^2) & &
\end{tikzcd} \tag{32}$$

where $\mathcal{Z}_+$ is the disk bundle over M defined in the manner of Section 3, $\mathcal{Z}^r_+ = p_2^{-1}(M \setminus S_\infty)$ is its restriction and $\mathcal{W}_+$ is the disk bundle over B defined in the manner of Section 2. Let $B \xleftarrow{p_1} \mathcal{W}_+ \xrightarrow{q_1} \mathbb{CP}^2$ be the double fibration for the standard Zoll projective structure on B.

Let $L_\mathbb{R}$ be the distribution on $\mathcal{W}_\mathbb{R}$ as in Section 2, and let $E_\mathbb{R}$ be the twistor distribution on $\mathcal{Z}_\mathbb{R}$ as in Section 3. Then the natural map $\Pi : \mathcal{Z}^r_+ \to \mathcal{W}_+$ is induced by the proof of Proposition 20, and Π is holomorphic on $\mathcal{Z}^r_+ \setminus \mathcal{Z}^r_\mathbb{R}$. We also have $\Pi_*(E_\mathbb{R}) = L_\mathbb{R}$ for the restriction $\Pi_\mathbb{R}$ of Π on $\mathcal{Z}^r_\mathbb{R}$. Since q_1 and q_2 are the maps which collapse the foliations defined by $L_\mathbb{R}$ and $E_\mathbb{R}$, Π induces a continuous map $\pi' : q_2(\mathcal{Z}^r_+) \to \mathbb{CP}^2$. We want to prove that π' smoothly extends to the natural projection $\pi : \mathbb{CP}^3 \setminus \{\zeta_0\} \to \mathbb{CP}^2$, and that $(\mathbb{CP}^1_\xi, P_\xi)$ is biholomorphic to $(\mathbb{CP}^1, \mathbb{RP}^1)$ for each $\xi \in \mathbb{RP}^2$, where $\mathbb{CP}^1_\xi = \pi^{-1}(\xi) \cup \{\zeta_0\}$ and $P_\xi = \mathbb{CP}^1_\xi \cap P$.

We study π' in more detail. Let α be an α-surface in $\mathcal{F}$, and let $C_\alpha = \alpha \cap S_\infty$. If we put $\alpha \setminus C_\alpha = \alpha_+ \sqcup \alpha_-$, then α_+ and α_- are two leaves of the α-surface foliation $\varpi : M \setminus S_\infty \to B$. If we put $b_\pm = \varpi(\alpha_\pm)$, then $\{b_+, b_-\}$ is the set of antipodal points on B by Proposition 29 and so on. So the corresponding holomorphic disks $D_{b_\pm} = q_1 \circ p_1^{-1}(b_\pm)$ have a common boundary, and $\mathbb{CP}^1_\alpha = D_{b_+} \cup D_{b_-}$ is a complex line in $\mathbb{CP}^2$. Then we obtain the following diagram as the restriction of (32):

$$\begin{tikzcd}
 & \mathcal{Z}_+|_\alpha \arrow[dl, "p^\alpha_2"'] \arrow[dr, "q^\alpha_2"] \arrow[d, phantom, "\cup"] & & & \\
\alpha \arrow[d, phantom, "\cup"] & \mathcal{Z}^r_+|_{\alpha \setminus C_\alpha} \arrow[dl] \arrow[d, "\Pi_\alpha"] \arrow[dr] & Q_\alpha \arrow[d, phantom, "\cup"] & & \\
\alpha \setminus C_\alpha \arrow[d, "\varpi_\alpha"] & D_{b_+} \sqcup D_{b_-} \arrow[dl] \arrow[dr] & q_2(\mathcal{Z}^r_+|_{\alpha \setminus C_\alpha}) \arrow[d, "\pi'_\alpha"] & \subset & Q_\alpha \setminus \{\zeta_0\} \arrow[dll, dotted, "\pi_\alpha"] \\
\{b_\pm\} & & \mathbb{CP}^1_\alpha & &
\end{tikzcd} \tag{33}$$

where $\mathcal{Z}_+|_\alpha = p_2^{-1}(\alpha)$, $\mathcal{Z}^r_+|_{\alpha \setminus C_\alpha} = p_2^{-1}(\alpha \setminus C\alpha)$, $Q_\alpha = q_2 \circ p_2^{-1}(\alpha)$, and so on.

Since C_α is a closed geodesic on S_∞ with respect to the standard Zoll projective structure, C_α has a natural involution which is the restriction of the involution on S_∞ exchanging the antipodal points. Hence $(\alpha, [\nabla], C_\alpha)$ defines an element of $\mathcal{M}'$, where $[\nabla]$ is the Zoll projective structure on α defined by Lemma 23.

Lemma 41. (i) $\mathcal{Z}_+^\circ|_\alpha = (\mathcal{Z}_+|_\alpha) \setminus (\mathcal{Z}_\mathbb{R}|_\alpha)$ *is a complex submanifold of* $\mathcal{Z}_+^\circ = \mathcal{Z}_+ \setminus \mathcal{Z}_\mathbb{R}$.(ii) *The double fibration* $\alpha \leftarrow \mathcal{Z}_+|_\alpha \rightarrow Q_\alpha$ *equals the double fibration for the Zoll projective structure on* α *given by Theorem 3. Consequently,* Q_α *is biholomorphic to* $\mathbb{CP}^2$.

Proof. Let $\mathcal{W}^\alpha = \mathbb{P}(T\alpha \otimes \mathbb{C})$, and we define $\mathcal{W}^\alpha_\pm$ as in Section 2, where $\mathcal{W}^\alpha = \mathcal{W}^\alpha_+ \cup \mathcal{W}^\alpha_-$. First we construct a diffeomorphism $\rho : \mathcal{W}^\alpha_+ \xrightarrow{\sim} \mathcal{Z}_+|_\alpha$. Let $x \in \alpha$ be any point, and take a null tetrad $\{e_0, e_1, \phi_0, \phi_1\}$ on an open neighborhood $U \subset M$ of x so that $T\alpha = \langle \phi_0, \phi_1 \rangle$. We can define diffeomorphisms $\mathcal{W}^\alpha|_{U\cap\alpha} \xleftarrow{\sim} \mathbb{CP}^1 \times (U \cap \alpha) \xrightarrow{\sim} \mathcal{Z}|_{U\cap\alpha}$ by using the trivializations of $\mathcal{W}^\alpha$ given by (1) and of $\mathcal{Z}$ given by (12). In other words, this map is characterized as the correspondence between a complex tangent line l of α and a complex β-plane β so that $l \subset \beta$, i.e.

$$\langle \phi_0 + \zeta\phi_1 \rangle \longleftrightarrow \langle e_0 + \zeta e_1, \phi_0 + \zeta\phi_1 \rangle .$$

This diffeomorphism does not depend on the choice of the null tetrad; hence we obtain a global diffeomorphism $\mathcal{W}^\alpha \xrightarrow{\sim} \mathcal{Z}|_\alpha$. We define ρ to be the restriction of this diffeomorphism on $\mathcal{W}^\alpha_+$.

We now check that $\mathcal{Z}_+^\circ|_\alpha$ is a complex submanifold of $\mathcal{Z}_+^\circ$. The complex structure on $\mathcal{Z}_+^\circ$ is defined so that the (0,1)-vector space is $K = \langle \mathfrak{m}_1, \mathfrak{m}_2, \bar\partial_\zeta \rangle$, where $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are the horizontal lifts of $e_0 + \zeta e_1$ and $\phi_0 + \zeta\phi_1$ respectively. On the other hand, the complex structure on $\mathcal{W}^{\alpha\,\circ}_+$ is defined so that the (0,1)-vector space is $K = \langle \mathfrak{n}, \bar\partial_\zeta \rangle$, where $\mathfrak{n}$ is the horizontal lift of $\phi_0 + \zeta\phi_1$. Then we obtain $\rho_*(\mathfrak{n}) = \mathfrak{m}_2$ and $\rho_*(\bar\partial_\zeta) = \bar\partial_\zeta$; hence ρ is holomorphic on the interior of $\mathcal{W}^\alpha_+$.

By a similar argument, we can check $(\rho_\mathbb{R})_*(L_\mathbb{R}) = E_\mathbb{R} \cap T\alpha$ for the restriction of ρ on $\mathcal{W}^\alpha_\mathbb{R}$. This means that $q : \mathcal{Z}_+|_\alpha \rightarrow Q_\alpha$ is the map which appears in the double fibration for the Zoll projective structure in the sense of Theorem 3. □

Lemma 42. $\mathbb{CP}^2 \simeq Q_\alpha \subset \mathbb{CP}^3$ *is a complex submanifold.*

Proof. Let $Q_{\alpha,\mathbb{R}} = q_2(\mathcal{Z}_\mathbb{R}|_\alpha)$. By Lemma 41, $Q_\alpha \setminus Q_{\alpha,\mathbb{R}} \subset \mathbb{CP}^3 \setminus P$ is a complex submanifold. So it is enough to check that, for each point $\zeta \in Q_{\alpha,\mathbb{R}}$, there is an open neighborhood in Q_α which is a complex submanifold of $\mathbb{CP}^3$.

Notice that $Q_{\alpha,\mathbb{R}} \subset P$ is a smooth real submanifold. This follows from the facts that $q^\alpha_{2,\mathbb{R}} : \mathcal{Z}_\mathbb{R}|_\alpha \rightarrow Q_{\alpha,\mathbb{R}}$ is an S^1-bundle, that $q_{2,\mathbb{R}} : \mathcal{Z}_\mathbb{R} \rightarrow P$ is an S^2-bundle, and that each fiber of $q^\alpha_{2,\mathbb{R}}$ is contained in some fiber of $q_{2,\mathbb{R}}$ as a smooth real submanifold.

We want to show that

$$T_\zeta Q_\alpha = T_\zeta Q_{\alpha,\mathbb{R}} \oplus J(T_\zeta Q_{\alpha,\mathbb{R}}) \tag{34}$$

for each $\zeta \in Q_{\alpha,\mathbb{R}}$, where J is the complex structure on $\mathbb{CP}^3$. Originally J is defined in the following manner (cf. [8], proof of Theorem 7.3). We can take a non-vanishing vector field u on $\mathcal{Z}_\mathbb{R}$ which spans $\ker(p_{2,\mathbb{R}})_*$ at every point. Moreover we can assume that $j(u)$ directs the interior of $\mathcal{Z}_+$ where j is the fiberwise complex structure of $\mathcal{Z}$ with respect to the $\mathbb{CP}^1$-bundle p_2. Then J is defined as the linear transform satisfying $J(q_{2*}(u)) = q_{2*}(j(u))$. Now Eq. (34) follows directly from this definition. □

Lemma 43. Π_α *satisfies the conditions* (Π1) *to* (Π3).

Proof. It is obvious that Π_α satisfies (Π1) and (Π3), so we check (Π2). Let $S_\infty/\mathbb{Z}_2$ be the set of pairs of antipodal points on S_∞, and we define a bijection $I : S_\infty/\mathbb{Z}_2 \rightarrow \mathbb{RP}^2$ by the following. For each $[x] \in S_\infty/\mathbb{Z}_2$, the set of closed geodesics through x defines a geodesic on $B = \tilde{\mathcal{G}}(S_\infty)$. Then we define $I([x]) \in \mathbb{RP}^2$ to be the point corresponding to this geodesic in the double fibration $B \xleftarrow{p_1} (\mathcal{W}_+, \mathcal{W}_\mathbb{R}) \xrightarrow{q_1} (\mathbb{CP}^2, \mathbb{RP}^2)$.

Since $C_\alpha = \alpha \cap S_\infty$, we can define $i_\alpha : C_\alpha/\mathbb{Z}_2 \rightarrow \mathbb{RP}^1_\alpha$ as the restriction of I. Then we have $i_\alpha([x]) \in \mathbb{RP}^1_\alpha$ from the definition, and we have $i_\alpha \circ \mu = q_1 \circ \Pi_{\alpha,\mathbb{R}}$. Actually, for example on α_+, each point $z \in \mathcal{Z}_\mathbb{R}|_{\alpha_+}$ corresponds to a pair (x, c) of a point $x \in \alpha_+$ and a closed geodesic c on α through x. Then we have $\mu(z) = [c \cap S_\infty]$ by definition. Hence $i_\alpha \circ \mu(z) = I([c \cap S_\infty])$. On the other hand, let β_c be the unique β-surface containing c; then

$\Pi_{\alpha,\mathbb{R}}(z) = \Pi_{\mathbb{R}}(z) \in \mathcal{W}_{\mathbb{R}}$ is the point defined by $(b_+, \varpi(\beta_c))$, where $b_+ = \varpi(x)$ and $\varpi(\beta_c)$ is a closed geodesic on B. Hence we have $q_1 \circ \Pi_{\alpha,\mathbb{R}}(z) = I([\beta_c \cap S_\infty])$ from the meaning of the double fibration for B. Since $c \cap S_\infty = \beta_c \cap S_\infty$, we obtain $i_\alpha \circ \mu = q_1 \circ \Pi_{\alpha,\mathbb{R}}$. $\square$

Corollary 44. *π'_α continuously and uniquely extends to $\pi_\alpha : Q_\alpha \setminus \{\zeta_0\} \to \mathbb{CP}^1$, and π_α is equivalent to the natural projection from ζ_0.*

Proof. Directly follows from Corollary 39. $\square$

Lemma 45. *There is a unique continuous extension $\pi : \mathbb{CP}^3 \setminus \{\zeta_0\} \to \mathbb{CP}^2$ of $\pi' : q_2(\mathcal{Z}^r_+) \to \mathbb{CP}^2$.*

Proof. Since $q_2(\mathcal{Z}^r_+)$ is dense in $\mathbb{CP}^3 \setminus \{\zeta_0\}$, the continuous extension is unique if it exists. So we prove the existence. For an element $\zeta \notin q_2(\mathcal{Z}^r_+)$, we define $\pi(\zeta)$ as follows. There is a unique $x \in S_\infty$ such that $\zeta \in D_x$, where $D_x = q_2 \circ p_2^{-1}(x)$. Let $\alpha \in \mathcal{F}$ be an α-surface through x; then $\zeta \in Q_\alpha$ and we put $\pi(\zeta) = \pi_\alpha(\zeta)$. Then $\pi(\zeta)$ does not depend on the choice of α, since $\pi_\alpha(\zeta) = i_\alpha([x]) = I(x)$ from the proof of Lemma 43.

Now we check that the above π is continuous. First, notice that $\cup_\alpha Q_\alpha = \mathbb{CP}^3$. Actually, for any $\zeta \in \mathbb{CP}^3$, we can take $x \in M$ so that $x \in p_2 \circ q_2^{-1}(\zeta)$, and if we take any $\alpha \in \mathcal{F}$ through x, then we obtain $\zeta \in Q_\alpha$. Since π is continuous on each Q_α, π is continuous on $\mathbb{CP}^3 \setminus \{\zeta_0\}$. $\square$

Lemma 46. *For each $\xi \in \mathbb{CP}^2$, $l_\xi = \pi^{-1}(\xi) \cup \{\zeta_0\}$ is a complex line in $\mathbb{CP}^3$. In consequence, $\pi : \mathbb{CP}^3 \setminus \{\zeta_0\} \to \mathbb{CP}^2$ is the projection.*

Proof. For each $\xi \in \mathbb{CP}^2$, there is at least one $\alpha \in \mathcal{F}$ such that $\xi \in \mathbb{CP}^1_\alpha$. Since $\pi^{-1}(\xi) = \pi_\alpha^{-1}(\xi)$ from the definition of π, $l_\xi = \pi_\alpha^{-1}(\xi) \cup \{\zeta_0\}$ is a complex line in $Q_\alpha \simeq \mathbb{CP}^2$ by Corollary 44. Moreover l_ξ is a rational curve in $\mathbb{CP}^3$ by Lemma 42.

Let $\xi' \in \mathbb{CP}^1_\alpha$ be a point different from ξ; then $l_{\xi'}$ is a rational curve in $\mathbb{CP}^3$. l_ξ and $l_{\xi'}$ are the complex lines in $Q_\alpha \simeq \mathbb{CP}^2$ which intersect only at ζ_0. Since $Q_\alpha \subset \mathbb{CP}^3$ is an embedding, l_ξ and $l_{\xi'}$ intersect only at ζ_0 in $\mathbb{CP}^3$, and the intersection is a node. Hence l_ξ and $l_{\xi'}$ are complex lines in $\mathbb{CP}^3$. $\square$

Proof (*Proof of* 40). For given $(M, [g], S_\infty, \mathcal{F})$, we already have a totally real embedding $\iota : \mathbb{RP}^3 \to \mathbb{CP}^3$ and $\zeta_0 \in P = \iota(\mathbb{RP}^3)$ which satisfies $\pi(P \setminus \{\zeta_0\}) = \mathbb{RP}^2$ for the standard projection $\pi : \mathbb{CP}^3 \setminus \{\zeta_0\} \to \mathbb{CP}^2$. For each $\xi \in \mathbb{RP}^2$, we put $\{x, \bar{x}\} = I^{-1}(\xi)$ which is the set of antipodal points of S_∞. Let D_x and $D_{\bar{x}}$ be the holomorphic disks corresponding to x and $\bar{x}$; then we have $l_\xi = \mathbb{CP}^1_\xi = D_x \cup D_{\bar{x}}$. Since $P_\xi = \partial D_x = \partial D_{\bar{x}}$, $(\mathbb{CP}^1_\xi, P_\xi)$ is biholomorphic to $(\mathbb{CP}^1, \mathbb{RP}^1)$. Hence (ι, ζ_0) is an element of $\mathcal{T}$ and this satisfies the required conditions in Theorem 32. $\square$

Next we prove the opposite direction of the main theorem.

Proposition 47. *Let $(\iota, \zeta_0) \in \mathcal{T}$ be an element which is contained in $f_{\mathcal{T}}^{-1}(V)$ in the terminology of Theorem 32. Then there is a unique element of $\mathcal{M}$ which satisfies the conditions in Theorem 32.*

Let $(\iota, \zeta_0) \in f_{\mathcal{T}}^{-1}(V)$, $P = \iota(\mathbb{RP}^3)$, and $\pi : (\mathbb{CP}^3 \setminus \{\zeta_0\}, P \setminus \{\zeta_0\}) \to (\mathbb{CP}^2, \mathbb{RP}^2)$ be the projection.

By Theorem 9, we have a space–time oriented self-dual Zollfrei conformal structure $(M, [g])$ and a double fibration $M \xleftarrow{p_2} \mathcal{Z}_+ \xrightarrow{q_2} \mathbb{CP}^3$ so that $q_{2,\mathbb{R}}(\mathcal{Z}_\mathbb{R}) = P$. Each point $x \in M$ corresponds to the holomorphic disk $D_x = q_2 \circ p_2^{-1}(x)$ in $(\mathbb{CP}^3, P)$, and each point $\zeta \in P$ corresponds to the β-surface $p_2 \circ q_2^{-1}(\zeta)$ on M. We define S_∞ to be the β-surface corresponding to the point $\zeta_0 \in P$. Notice that, for each $x \in M \setminus S_\infty$, we obtain $\zeta_0 \notin D_x$, i.e. D_x is a holomorphic disk in $(\mathbb{CP}^3 \setminus \{\zeta_0\}, P \setminus \{\zeta_0\})$.

Let $B \xleftarrow{p_1} (\mathcal{W}_+, \mathcal{W}_\mathbb{R}) \xrightarrow{q_1} (\mathbb{CP}^2, \mathbb{RP}^2)$ be the double fibration given by Theorem 3. Each point in B corresponds to some holomorphic disk in $(\mathbb{CP}^2, \mathbb{RP}^2)$, and B is equipped with the standard Zoll projective structure.

Let $B/\mathbb{Z}_2$ be the set of pairs of antipodal points in B. Let $b \in B/\mathbb{Z}_2$ be a pair of antipodal points $\{b_+, b_-\}$; then the corresponding holomorphic disks $D_{b_\pm}$ have a common boundary, so $\mathbb{CP}^1_b = D_{b_+} \cup D_{b_-}$ is a complex line in $\mathbb{CP}^2$. If we put $Q_b = \pi^{-1}(\mathbb{CP}^1_b) \cup \{\zeta_0\}$, then Q_b is a complex plane in $\mathbb{CP}^3$, since π is the projection. We put $N_b = P \cap Q_b$.

Lemma 48. *(Q_b, N_b) and ζ_0 define an element of $\mathcal{T}_0$.*

Proof. N_b is the one-point compactification of $\pi_{\mathbb{R}}^{-1}(\mathbb{RP}^1_b)$, where $\mathbb{RP}^1_b = \mathbb{RP}^2 \cap \mathbb{CP}^1_b$ and $\pi_{\mathbb{R}} : P \setminus \{\zeta_0\} \to \mathbb{RP}^2$ is the restriction of π. Since $\pi_{\mathbb{R}}$ is a non-trivial $\mathbb{R}$-fibration, N_b is an embedded $\mathbb{RP}^2$ in Q_b. Since $\zeta_0 \in N_b$, and since the second condition in Definition 35 obviously holds, (N_b, ζ_0) defines an element of $\mathcal{T}_0$. $\square$

From Lemma 48, we obtain a diagram similar to (30):

$$\begin{tikzcd}
 & \mathcal{W}_+(b) \arrow[ddl, "p_1(b)"'] \arrow[ddr, "q_1(b)"] & \\
 & \cup & \\
\alpha(b) & \mathcal{W}^r_+(b) \arrow[ddl] \arrow[dd, "\Pi_b"] \arrow[ddr] & Q_b \\
\cup & & \cup \\
\alpha(b) \setminus C_b \arrow[d, "\varpi_b"] & D_{b_+} \sqcup D_{b_-} \arrow[dl] \arrow[dr] & Q_b \setminus \{\zeta_0\} . \arrow[d, "\pi"] \\
\{b_\pm\} & & \mathbb{CP}^1_b
\end{tikzcd} \tag{35}$$

Lemma 49. *There is a natural injection $\alpha(b) \to M$. Moreover there is a smooth map $\varpi : M \setminus S_\infty \to B$ such that the restriction of ϖ on $\alpha(b)$ is equal to ϖ_b.*

Proof. Let $p \in \alpha(b)$ be a point and $D_p = q_1(b) \circ p_1(b)^{-1}(p)$ be the corresponding holomorphic disk in (Q_b, N_b). Let $\mathcal{L}_b = \{D_p\}_{p \in \alpha(\beta)}$ be the family of such holomorphic disks in (Q_b, N_b); then $\mathcal{L}_b$ foliates $Q_b \setminus N_b$. We will soon show that $\cup_b \mathcal{L}_b$ defines a family of holomorphic disks in $(\mathbb{CP}^3, P)$ foliating $\mathbb{CP}^3 \setminus P$; then it follows that $\alpha(b)$ is a subset of the moduli space M of holomorphic disks. Moreover, ϖ is naturally induced as the map between the sets of holomorphic disks, so this is smooth and $\varpi|_{\alpha(b)} = \varpi_b$.

Now we prove that $\cup_b \mathcal{L}_b$ foliates $\mathbb{CP}^3 \setminus P$. For distinct points $b, b' \in B/\mathbb{Z}_2$, $\mathbb{CP}^1_b \cap \mathbb{CP}^1_{b'}$ consists of one point $\xi \in \mathbb{RP}^2$. Then $Q_b \cap Q'_b = \pi^{-1}(\xi) \cup \{\zeta_0\} = \mathbb{CP}^1_\xi$. If we put $P_\xi = \mathbb{CP}^1_\xi \cap P$, then $(\mathbb{CP}^1_\xi, P_\xi)$ is biholomorphic to $(\mathbb{CP}^1, \mathbb{RP}^1)$ by definition. So we can write $\mathbb{CP}^1_\xi = D_1 \cup D_2$, where $\partial D_1 = \partial D_2 = P_\xi$. As in the proof of Theorem 37, the D_i $(i = 1, 2)$ are contained in $\mathcal{L}_b$ and $\mathcal{L}_{b'}$. Hence $\mathcal{L}_b \cup \mathcal{L}_{b'}$ foliates $(Q_b \cup Q_{b'}) \setminus (N_b \cup N_{b'})$. Since $\cup_b Q_b = \mathbb{CP}^3$, it follows that $\cup_b \mathcal{L}_b$ foliates $\mathbb{CP}^3 \setminus P$. $\square$

It follows from Lemma 49 that the diagram (35) is the restriction of the diagram (32), i.e. $\mathcal{W}_+(b) = q_2^{-1}(Q_b) = p_2^{-1}(\alpha(b))$, $p_1(b) = p_2|_{\mathcal{W}_+(b)} = q_2|_{\mathcal{W}_+(b)}$ and so on. Now we put $\mathcal{F} = \{\alpha(b)\}_{b \in B/\mathbb{Z}_2}$ which is a family of embedded 2-spheres in M. Each $\alpha(b)$ has a Zoll projective structure defined by Theorem 37 and Lemma 48.

Lemma 50. *$\alpha(b)$ is a closed α-surface.*

Proof. Each closed geodesic on $\alpha(b)$ is written in the form

$$C(\zeta) = p_1(b) \circ q_1(b)^{-1}(\zeta)$$

for some $\zeta \in N_b$, while each β-surface is written in the form $\beta(\zeta) = p_2 \circ q_2^{-1}(\zeta)$ for some $\zeta \in P$. Hence each closed geodesic on $\alpha(b)$ is contained in some β-surface. So $\alpha(b)$ is totally null. Since a totally null surface is either an α-surface or a β-surface, and since $\alpha(b)$ is not a β-surface, this is an α-surface. $\square$

Proof (*Proof of* 47). For given (ι, ζ_0), we take $(M, [g], S_\infty, \mathcal{F})$ as above. For each $\alpha(b) \in \mathcal{F}$, $\alpha(b) \cap S_\infty = C_b$ is always non-empty. $\mathcal{F}$ defines a smooth foliation on $M \setminus S_\infty$; hence $(M, [g], S_\infty, \mathcal{F})$ is an element of $\mathcal{M}$. This element satisfies the conditions in Theorem 32. $\square$

Theorem 32 follows from Propositions 40 and 47.

Acknowledgements

The author most gratefully thanks his supervisor Mikio Furuta, for much advice and continuous help. This research is supported by the 21st century COE program at the Graduate School of Mathematical Sciences, the University of Tokyo.

Appendix A. Self-dual foliation

In Section 5, we argued on a basic α-surface foliation, while Calderbank considered a self-dual α-surface foliation in [2]. Here we check that these conditions are equivalent in the assumption of the self-duality condition of the metric.

Let (M, g) be a 4-manifold with a neutral metric and let S^- be the negative spin bundle; then an α-plane distribution on M one-to-one corresponds with a subbundle $l : L \to S^-$.

If we fix $l : L \to S^-$ and take a connection ∇ on L, then we have the covariant derivative operator $D^\nabla : \Gamma(S^- \otimes L^*) \to \Gamma(TM \otimes S^- \otimes L^*)$. Noticing the identification $T^*M \cong S^{+*} \otimes S^{-*} \cong S^+ \otimes S^-$, we put $T^*M \odot S^- = S^+ \otimes (S^- \odot S^-)$, where $S^- \odot S^-$ is the symmetric tensor. We obtain the twistor operator $\mathcal{T}^\nabla : \Gamma(S^- \otimes L^*) \to \Gamma(TM \odot S^- \otimes L^*)$ by composing the symmetrization with the covariant derivative D^∇.

Definition 51 (*Cf. [2]*). A connection ∇ on L is called canonical if and only if it satisfies $\mathcal{T}^\nabla l = 0$. An α-surface foliation ϖ is called self-dual if and only if, for the corresponding subbundle $l : L \to S^-$, (i) there is a canonical connection ∇ on L, and (ii) ∇ is self-dual.

The following property is explained in [2]; however we give the proof again by using an explicit description.

Proposition 52. *Let $l : L \to S_-$ be a subbundle; then the α-surface distribution corresponding to l is integrable if and only if the canonical connection on L exists. The canonical connection is unique if it exists.*

Proof. Since the conditions are entirely local, we can assume that $S^- = M \times \mathbb{R}^2$ and $L = M \times \mathbb{R}$ are trivial bundles, and that $l : L \to S^-$ is a constant section $l = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \Gamma(S^- \otimes L^*)$. Let $\begin{pmatrix} e_0 & \phi_0 \\ e_1 & \phi_1 \end{pmatrix}$ be a null tetrad respecting the trivialization of S^-; then the α-plane distribution corresponding to l is given by $\langle \phi_0, \phi_1 \rangle$.

We denote the Levi-Civita connection of g in the same way as in (10). Let ∇ be a connection on L represented by a connection 1-form τ; then the equation $\mathcal{T}^\nabla l = 0$ is decomposed into the following equations:

$$\begin{cases} \left(\dfrac{a+d}{2} + \tau\right)(e_A) = 0, \\ e(e_A) + \left(\dfrac{a+d}{2} + \tau\right)(\phi_A) = 0, \\ e(\phi_A) = 0, \end{cases} \qquad (A = 0, 1). \tag{36}$$

So the canonical connection on L exists if and only if $e(\phi_0) = e(\phi_1) = 0$. This holds if and only if the α-plane distribution $\langle \phi_0, \phi_1 \rangle$ is integrable as in (18). The uniqueness of the canonical connection is obvious from (36). $\square$

Lemma 53. *Let (M, g) be a 4-manifold with a neutral metric g and $\varpi : M \to B$ be an α-surface foliation. Then ϖ is self-dual if and only if the following equations hold:*

$$\begin{aligned} &\phi_0(a+d)(e_0) = \phi_1(a+d)(e_1) = 0, \\ &\phi_0 a(e_1) + \phi_1 a(e_0) = -(\phi_0 d(e_1) + \phi_1 d(e_0)). \end{aligned} \tag{37}$$

Proof. Take a null tetrad as in Proposition 10. Since this null tetrad fits with the proof of Proposition 52, the canonical connection is defined by the 1-form τ satisfying (36). This connection is self-dual if and only if

$$d\tau(e_0 \wedge \phi_0) = d\tau(e_1 \wedge \phi_1) = d\tau(e_0 \wedge \phi_1 + e_1 \wedge \phi_0) = 0,$$

and it is equivalent to (37), since $e = 0$ by Lemma 13. $\square$

Proposition 54. *Let (M, g) be a 4-manifold with a neutral self-dual metric and $\varpi : M \to B$ be an α-surface foliation. Then ϖ is self-dual if and only if it is basic.*

Proof. We take the coordinate as in Proposition 15, and define g as the form (23). If we take a null tetrad as in (25), then each element of the connection form is given by (37). Noticing (24), ϖ is self-dual if and only if

$$\partial_2\partial_3 p = \partial_2\partial_3 q = \partial_3^2 p = \partial_2^2 q = \partial_2^2 r = \partial_2^2 r = 0. \tag{38}$$

This is equivalent to the basic condition (27). □

Appendix B. Null conformal Killing vector field

Here we treat the case of Dunajski and West, i.e. the case when there is a null conformal Killing vector field on (M, g). Our method is a little far from the general treatment of twistor theory (cf. [13]), but the calculations are easier.

Definition 55. Let (M, g) be a 4-manifold with a neutral metric; then a vector field K on M is called the conformal Killing vector field when there is a function η on M satisfying $\mathcal{L}_K(g) = \eta g$.

Notice that this condition depends only on the conformal structure on M.

Proposition 56 ([4]). *Let K be a null conformal Killing vector field on $(M, [g])$; then there is a unique α-plane distribution and a unique β-plane distribution on M which contains K, and these distributions are both integrable.*

For the proof of this, see [4] Lemma 1, and the remark following it.

Let (M, g) be as above and K be a null conformal Killing vector field on M. From Proposition 56, M has an α-surface foliation. Taking M smaller, we can assume that the leaf space B of this α-surface foliation is a two-dimensional manifold. Then we can take the coordinates and the null tetrad as in Proposition 10. Now, since K is a section of the α-surface distribution, we can write

$$K = K^0\phi_0 + K^1\phi_1 \tag{39}$$

with some functions K^0 and K^1. We use the same description for the Levi-Civita connection as in (10), and, for simplicity, we write $a(e_i) = a_i$ and so on.

Lemma 57. *A null vector field $K = K^0\phi_0 + K^1\phi_1$ is a conformal Killing vector field if and only if there is a function η on M and the following conditions hold:*

$$\begin{cases} \phi_0 K^1 = \phi_1 K^0 = 0 \\ \phi_0 K^0 = \phi_1 K^1 = \eta \\ e_0 K^1 + b_0 K^0 - a_0 K^1 = 0 \\ e_1 K^0 - d_1 K^0 + b_1 K^1 = 0 \\ e_0 K^0 - d_0 K^0 + b_0 K^1 = e_1 K^1 + b_1 K^0 - a_1 K^1. \end{cases} \tag{40}$$

Proof. Direct calculation from $\mathcal{L}_K g = \eta g$. □

Lemma 58. *Let $(M, [g])$ be a neutral self-dual conformal structure, and K be a null conformal Killing vector field on M; then*

$$\phi_0 a_1 + \phi_1 a_0 = \phi_0 d_1 + \phi_1 d_0 = 0, \qquad \phi_0 a_0 = \phi_1 d_1 = 0, \qquad \phi_0 b_1 = \phi_1 b_0 = 0. \tag{41}$$

Proof. Differentiating (40), and using (17), we have

$$\begin{cases} e_0\eta = -(\phi_1 b_0)K^0 + (\phi_1 a_0)K^1, \\ e_1\eta = (\phi_0 d_1)K^0 - (\phi_0 b_1)K^1, \\ e_0\eta = (\phi_0(d_0 + b_1))K^0 - (\phi_0 a_1)K^1, \\ e_1\eta = -(\phi_1 d_0)K^0 + (\phi_1(a_1 + b_0))K^1. \end{cases} \tag{42}$$

Comparing these equations, and from (22), we obtain the first and the second equation of (41). Operating with ϕ_0 on the first of (42), and with ϕ_1 on the second, we have $(\phi_1 b_0)\eta = (\phi_0 b_1)\eta = 0$. We can assume $\eta \neq 0$ by changing the metric in the conformal class $[g]$, so we have $\phi_0 b_1 = \phi_1 b_0 = 0$. □

Theorem 59. *The α-plane distribution defined by Proposition 56 is basic.*

Proof. Condition (27) is obtained directly from (41). Hence this distribution is basic. □

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